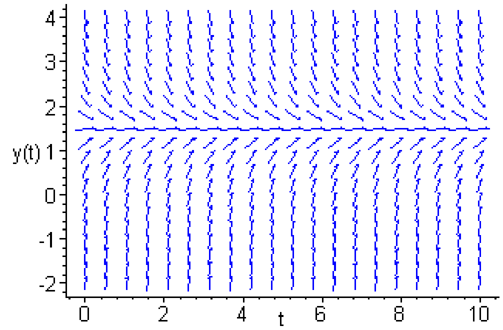
Chapter One

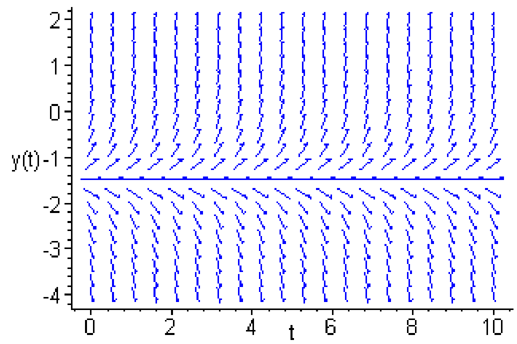
Section 1.1

1.



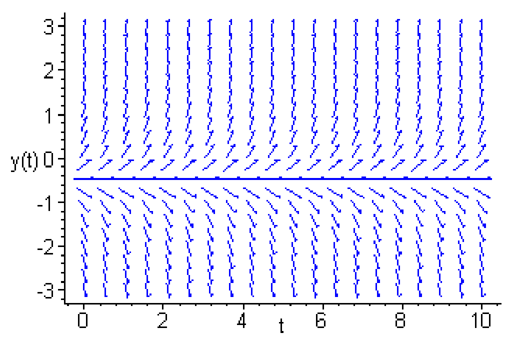
For $y > 1.5$, the slopes are *negative*, and hence the solutions decrease. For $y < 1.5$, the slopes are *positive*, and hence the solutions increase. The equilibrium solution appears to be $y(t) = 1.5$, to which all other solutions converge.

3.



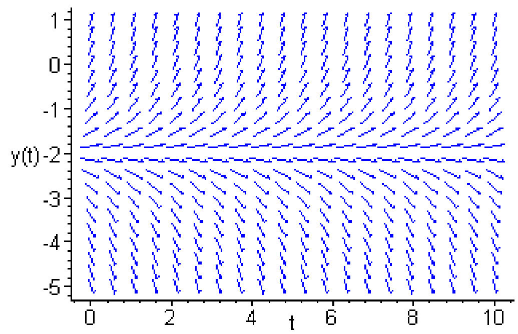
For $y > -1.5$, the slopes are *positive*, and hence the solutions increase. For $y < -1.5$, the slopes are *negative*, and hence the solutions decrease. All solutions appear to diverge away from the equilibrium solution $y(t) = -1.5$.

5.



For $y > -1/2$, the slopes are *positive*, and hence the solutions increase. For $y < -1/2$, the slopes are *negative*, and hence the solutions decrease. All solutions diverge away from the equilibrium solution $y(t) = -1/2$.

6.



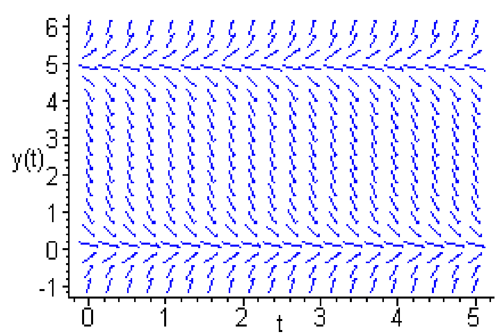
For $y > -2$, the slopes are *positive*, and hence the solutions increase. For $y < -2$, the slopes are *negative*, and hence the solutions decrease. All solutions diverge away from
the equilibrium solution $y(t) = -2$.

8. For *all* solutions to approach the equilibrium solution $y(t) = 2/3$, we must have $y' < 0$ for $y > 2/3$, and $y' > 0$ for $y < 2/3$. The required rates are satisfied by the differential equation $y' = 2 - 3y$.

9. For solutions *other* than $y(t) = 2$ to diverge from $y = 2$, $y(t)$ must be an *increasing* function for $y > 2$, and a *decreasing* function for $y < 2$. The simplest differential equation
whose solutions satisfy these criteria is $y' = y - 2$.

10. For solutions *other* than $y(t) = 1/3$ to diverge from $y = 1/3$, we must have $y' < 0$ for $y < 1/3$, and $y' > 0$ for $y > 1/3$. The required rates are satisfied by the differential equation $y' = 3y - 1$.

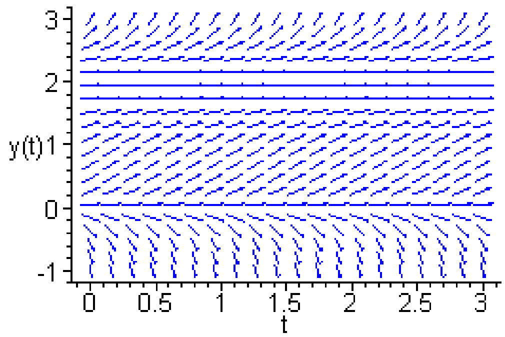
12.



Note that $y' = 0$ for $y = 0$ and $y = 5$. The two equilibrium solutions are $y(t) = 0$ and $y(t) = 5$. Based on the direction field, $y' > 0$ for $y > 5$; thus solutions with initial values *greater* than 5 diverge from the solution $y(t) = 5$. For $0 < y < 5$, the slopes are *negative*, and hence solutions with initial values *between* 0 and 5 all decrease toward the

solution $y(t) = 0$. For $y < 0$, the slopes are all *positive*; thus solutions with initial values
less than 0 approach the solution $y(t) = 0$.

14.

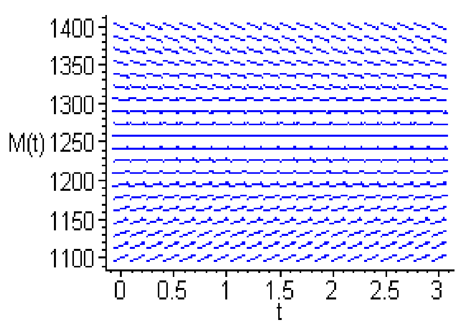


Observe that $y' = 0$ for $y = 0$ and $y = 2$. The two equilibrium solutions are $y(t) = 0$ and $y(t) = 2$. Based on the direction field, $y' > 0$ for $y > 2$; thus solutions with initial values *greater* than 2 diverge from $y(t) = 2$. For $0 < y < 2$, the slopes are also *positive*, and hence solutions with initial values *between* 0 and 2 all increase toward the solution
$y(t) = 2$. For $y < 0$, the slopes are all *negative*; thus solutions with initial values *less* than 0 diverge from the solution $y(t) = 0$.

16. (a) Let $M(t)$ be the total amount of the drug (*in milligrams*) in the patient's body at any
given time t (hrs). The drug is administered into the body at a *constant* rate of 500 mg/hr.
The rate at which the drug *leaves* the bloodstream is given by $0.4M(t)$. Hence the accumulation rate of the drug is described by the differential equation

$$\frac{dM}{dt} = 500 - 0.4\,M \quad (mg/hr).$$

(b)



Based on the direction field, the amount of drug in the bloodstream approaches the equilibrium level of $1250\,mg$ $(within\,a\,few\,hours)$.

18. (a) Following the discussion in the text, the differential equation is

$$m\frac{dv}{dt} = mg - \gamma v^2$$

or equivalently,

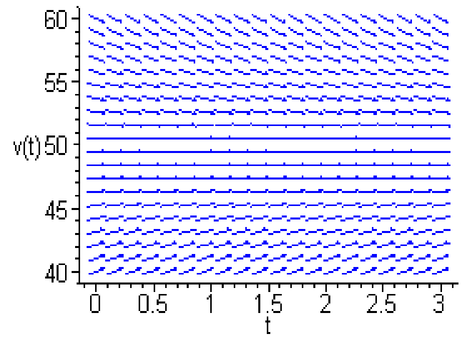
$$\frac{dv}{dt} = g - \frac{\gamma}{m}v^2 .$$

(b) After a long time, $\frac{dv}{dt} \approx 0$. Hence the object attains a *terminal velocity* given by

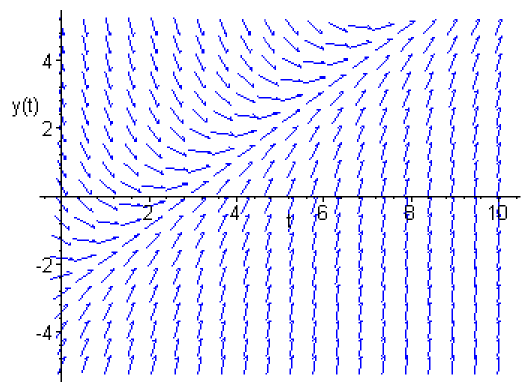
$$v_\infty = \sqrt{\frac{mg}{\gamma}} .$$

(c) Using the relation $\gamma v_\infty^2 = mg$, the required *drag coefficient* is $\gamma = 0.0408\, kg/sec$.

(d)

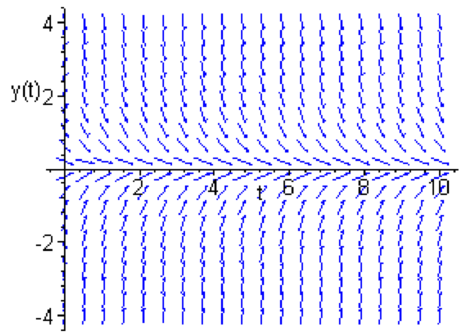


19.



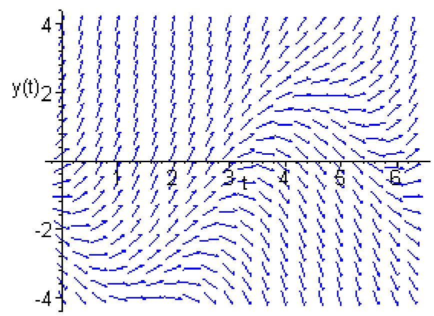
All solutions appear to approach a linear asymptote $(with\, slope\, equal\, to\, 1)$. It is easy to verify that $y(t) = t - 3$ is a solution.

20.



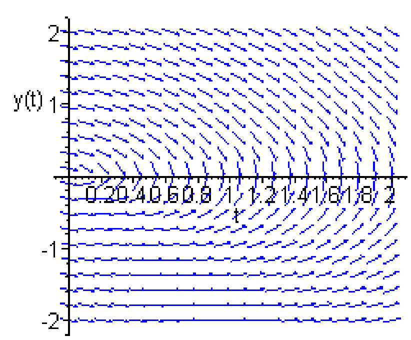
All solutions approach the equilibrium solution $y(t) = 0$.

23.



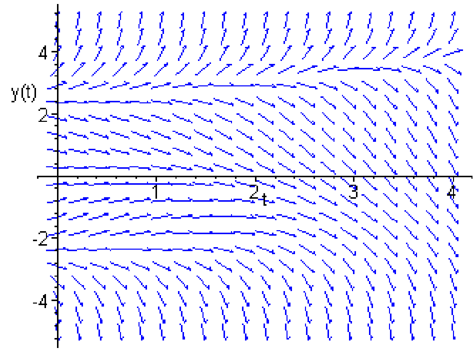
All solutions appear to *diverge* from the sinusoid $y(t) = -\frac{3}{\sqrt{2}} sin(t + \frac{\pi}{4}) - 1$,
which is also a solution corresponding to the initial value $y(0) = -5/2$.

25.

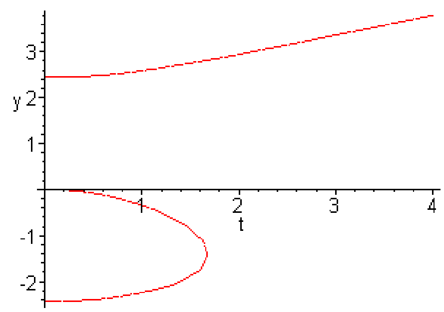


All solutions appear to converge to $y(t) = 0$. First, the rate of change is small. The slopes
eventually increase very rapidly in *magnitude*.

26.



The direction field is rather complicated. Nevertheless, the collection of points at which the slope field is *zero*, is given by the implicit equation $y^3 - 6y = 2t^2$. The graph of these points is shown below:



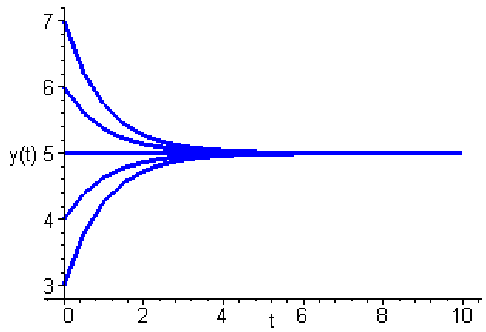
The *y-intercepts* of these curves are at $y = 0\,, \pm\sqrt{6}$. It follows that for solutions with initial values $y > \sqrt{6}$, all solutions increase without bound. For solutions with initial values in the range $y < -\sqrt{6}$ *and* $0 < y < \sqrt{6}$, the slopes remain *negative*, and hence
these solutions decrease without bound. Solutions with initial conditions in the range $-\sqrt{6} < y < 0$ initially increase. Once the solutions reach the critical value, given by the equation $y^3 - 6y = 2t^2$, the slopes become negative and *remain* negative. These solutions eventually decrease without bound.

Section 1.2

$1(a)$ The differential equation can be rewritten as

$$\frac{dy}{5-y} = dt\,.$$

Integrating both sides of this equation results in $-ln|5-y| = t + c_1$, or equivalently, $5 - y = c\,e^{-t}$. Applying the initial condition $y(0) = y_0$ results in the specification of the constant as $c = 5 - y_0$. Hence the solution is $y(t) = 5 + (y_0 - 5)e^{-t}$.

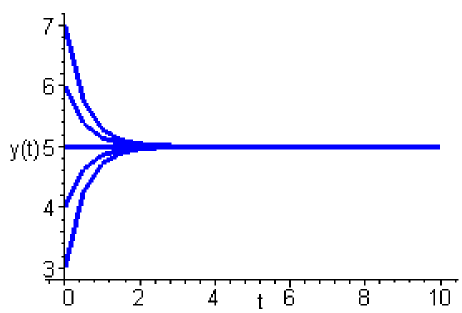


All solutions appear to converge to the equilibrium solution $y(t) = 5$.

$1(c)$. Rewrite the differential equation as

$$\frac{dy}{10-2y} = dt\,.$$

Integrating both sides of this equation results in $-\frac{1}{2}ln|10-2y| = t + c_1$, or equivalently,
$5 - y = c\,e^{-2t}$. Applying the initial condition $y(0) = y_0$ results in the specification of the constant as $c = 5 - y_0$. Hence the solution is $y(t) = 5 + (y_0 - 5)e^{-2t}$.

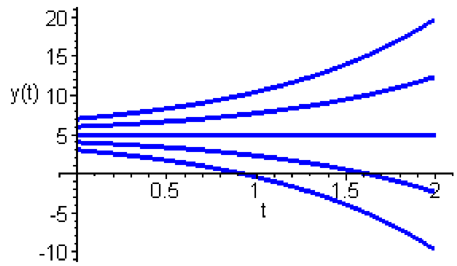


All solutions appear to converge to the equilibrium solution $y(t) = 5$, but at a *faster* rate than in Problem 1a .

$2(a)$. The differential equation can be rewritten as

$$\frac{dy}{y-5} = dt\,.$$

Integrating both sides of this equation results in $ln|y-5| = t + c_1$, or equivalently, $y - 5 = c\,e^t$. Applying the initial condition $y(0) = y_0$ results in the specification of the constant as $c = y_0 - 5$. Hence the solution is $y(t) = 5 + (y_0 - 5)e^t$.

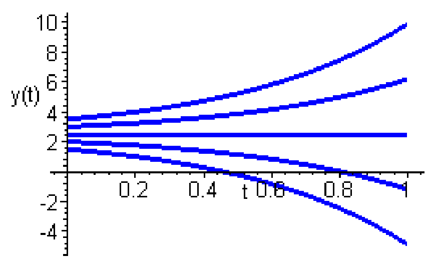


All solutions appear to diverge from the equilibrium solution $y(t) = 5$.

$2(b)$. Rewrite the differential equation as

$$\frac{dy}{2y-5} = dt\,.$$

Integrating both sides of this equation results in $\frac{1}{2}ln|2y-5| = t + c_1$, or equivalently, $2y - 5 = c\,e^{2t}$. Applying the initial condition $y(0) = y_0$ results in the specification of the constant as $c = 2y_0 - 5$. Hence the solution is $y(t) = 2.5 + (y_0 - 2.5)e^{2t}$.

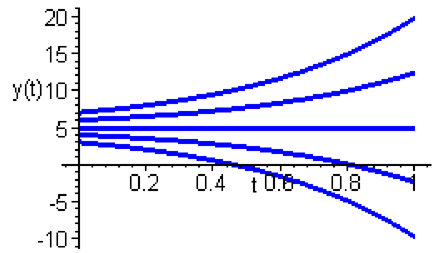


All solutions appear to diverge from the equilibrium solution $y(t) = 2.5$.

$2(c)$. The differential equation can be rewritten as

$$\frac{dy}{2y-10} = dt\,.$$

Integrating both sides of this equation results in $\frac{1}{2}ln|2y-10| = t + c_1$, or equivalently, $y - 5 = c\,e^{2t}$. Applying the initial condition $y(0) = y_0$ results in the specification of the constant as $c = y_0 - 5$. Hence the solution is $y(t) = 5 + (y_0 - 5)e^{2t}$.



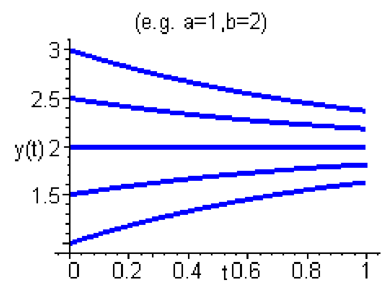
All solutions appear to diverge from the equilibrium solution $y(t) = 5$.

$3(a)$. Rewrite the differential equation as

$$\frac{dy}{b - ay} = dt\,,$$

which is valid for $y \neq b\,/a$. Integrating both sides results in $\frac{-1}{a} ln|b - ay| = t + c_1$, or equivalently, $b - ay = c\,e^{-at}$. Hence the general solution is $y(t) = (b - c\,e^{-at})/a$. Note that if $y = b/a$, then $dy/dt = 0$, and $y(t) = b/a$ is an equilibrium solution.

(b)



(i) As a increases, the equilibrium solution gets closer to $y(t) = 0$, from above. Furthermore, the *convergence rate* of all solutions, that is, a, also increases.
(ii) As b increases, then the equilibrium solution $y(t) = b/a$ also becomes larger. In this case, the convergence rate remains the same.
(iii) If a and b both increase (*but* $b/a =$ *constant*), then the equilibrium solution $y(t) = b/a$ remains the same, but the *convergence rate* of all solutions increases.

$5(a)$. Consider the simpler equation $dy_1/dt = -\,ay_1$. As in the previous solutions, rewrite the equation as

$$\frac{dy_1}{y_1} = -\,a\,dt\,.$$

Integrating both sides results in $y_1(t) = c\,e^{-at}$.
(b). Now set $y(t) = y_1(t) + k$, and substitute into the original differential equation. We find that

$$-\,ay_1 + 0 = \,-\,a(y_1 + k) + b\,.$$

That is, $-\,ak + b = 0$, and hence $k = b/a$.
(c). The general solution of the differential equation is $y(t) = c\,e^{-at} + b/a$. This is exactly the form given by Eq. (17) in the text. Invoking an initial condition $y(0) = y_0$, the solution may also be expressed as $y(t) = b/a + (y_0 - b/a)e^{-at}$.

$6(a)$. The general solution is $p(t) = 900 + c\,e^{t/2}$, that is, $p(t) = 900 + (p_0 - 900)e^{t/2}$. With $p_0 = 850$, the specific solution becomes $p(t) = 900 - 50e^{t/2}$. This solution is a *decreasing* exponential, and hence the time of extinction is equal to the number of months
it takes, say t_f, for the population to reach *zero*. Solving $900 - 50e^{t_f/2} = 0$, we find that $t_f \ = 2\,ln(900/50) = 5.78$ *months*.
(b) The solution, $p(t) = 900 + (p_0 - 900)e^{t/2}$, is a *decreasing* exponential as long as $p_0 < 900$. Hence $900 + (p_0 - 900)e^{t_f/2} = 0$ has only *one* root, given by

$$t_f \ = 2\,ln\left(\frac{900}{900 - p_0}\right).$$

(c). The answer in part (b) is a general equation relating time of extinction to the value of
the initial population. Setting $t_f = 12$ *months*, the equation may be written as

$$\frac{900}{900 - p_0} = e^6,$$

which has solution $p_0 = 897.7691$. Since p_0 is the initial population, the appropriate answer is $p_0 = 898$ *mice*.

$7(a)$. The general solution is $p(t) = p_0\,e^{rt}$. Based on the discussion in the text, time t is measured in *months*. Assuming 1 *month* $= 30$ *days*, the hypothesis can be expressed as $p_0\,e^{r\cdot 1} = 2p_0$. Solving for the rate constant, $r = ln(2)$, with units of *per month*.

(b). N *days* $= N/30$ *months*. The hypothesis is stated mathematically as $p_0 e^{rN/30} = 2p_0$
.
It follows that $rN/30 = ln(2)$, and hence the rate constant is given by $r = 30\,ln(2)/N$. The units are understood to be *per month*.

$9(a)$. Assuming *no air resistance*, with the positive direction taken as *downward*, Newton's
Second Law can be expressed as

$$m\frac{dv}{dt} = mg$$

in which g is the *gravitational constant* measured in appropriate units. The equation can be written as $dv/dt = g$, with solution $v(t) = gt + v_0$. The object is released with an initial velocity v_0.

(b). Suppose that the object is released from a height of h *units* above the ground. Using the fact that $v = dx/dt$, in which x is the *downward displacement* of the object, we obtain the differential equation for the displacement as $dx/dt = gt + v_0$. With the origin placed at the point of release, direct integration results in $x(t) = gt^2/2 + v_0\,t$. Based on the chosen coordinate system, the object reaches the ground when $x(t) = h$. Let $t = T$ be the time that it takes the object to reach the ground. Then $gT^2/2 + v_0 T = h$. Using the quadratic formula to solve for T,

$$T = \frac{-\,v_0 \pm \sqrt{v_0 + 2gh}}{g}\ .$$

The *positive* answer corresponds to the time it takes for the object to fall to the ground. The *negative* answer represents a previous instant at which the object could have been launched upward (*with the same impact speed*), only to ultimately fall downward with speed v_0, from a height of h *units* above the ground.

(c). The impact speed is calculated by substituting $t = T$ into $v(t)$ in part (a). That is, $v(T) = \sqrt{v_0 + 2gh}$.

10$(a$,$b)$. The general solution of the differential equation is $Q(t) = c\,e^{-rt}$. Given that $Q(0) = 100$ *mg*, the value of the constant is given by $c = 100$. Hence the amount of thorium-234 present at any time is given by $Q(t) = 100\,e^{-rt}$. Furthermore, based on the hypothesis, setting $t = 1$ results in $82.04 = 100\,e^{-r}$. Solving for the rate constant, we find that $r = -\,ln(82.04/100) = .19796$/*week* or $r = .02828$/*day*.

(c). Let T be the time that it takes the isotope to decay to *one-half* of its original amount. From part (a), it follows that $50 = 100\,e^{-rT}$, in which $r = .19796$/*week*. Taking the natural logarithm of both sides, we find that $T = 3.5014$ *weeks* or $T = 24.51$ *days*.

11. The general solution of the differential equation $dQ/dt = -\,r\,Q$ is $Q(t) = Q_0 e^{-rt}$, in which $Q_0 = Q(0)$ is the initial amount of the substance. Let τ be the time that it takes the substance to decay to *one-half* of its original amount, Q_0. Setting $t = \tau$ in the solution, we have $0.5\,Q_0 = Q_0 e^{-r\tau}$. Taking the natural logarithm of both sides, it follows that $-\,r\tau = ln(0.5)$ or $r\tau = ln\,2$.

12. The differential equation governing the amount of radium-226 is $dQ/dt = -r\,Q$, with solution $Q(t) = Q(0)e^{-rt}$. Using the result in Problem 11, and the fact that the half-life $\tau = 1620$ *years*, the decay rate is given by $r = ln(2)/1620$ *per year*. The amount of radium-226, after t years, is therefore $Q(t) = Q(0)e^{-0.00042786t}$. Let T be the time that it takes the isotope to decay to $3/4$ of its original amount. Then setting $t = T$,
and $Q(T) = \frac{3}{4}Q(0)$, we obtain $\frac{3}{4}Q(0) = Q(0)e^{-0.00042786T}$. Solving for the decay time, it follows that $-0.00042786\,T = ln(3/4)$ or $T = 672.36$ *years*.

13. The solution of the differential equation, with $Q(0) = 0$, is
$Q(t) = CV(1 - e^{-t/CR})$.
As $t \to \infty$, the exponential term vanishes, and hence the limiting value is $Q_L = CV$.

14(a). The *accumulation* rate of the chemical is $(0.01)(300)$ *grams per hour*. At any given time t, the *concentration* of the chemical in the pond is $Q(t)/10^6$ *grams per gallon*.
Consequently, the chemical *leaves* the pond at a rate of $(3 \times 10^{-4})Q(t)$ *grams per hour*. Hence, the rate of change of the chemical is given by

$$\frac{dQ}{dt} = 3 - 0.0003\,Q(t) \;\; gm/hr\,.$$

Since the pond is initially free of the chemical, $Q(0) = 0$.

(b). The differential equation can be rewritten as

$$\frac{dQ}{10000 - Q} = 0.0003\,dt\,.$$

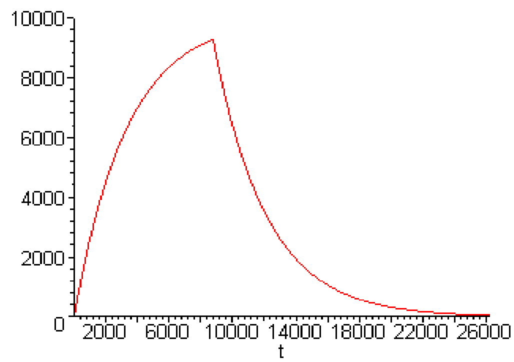
Integrating both sides of the equation results in $-ln|10000 - Q| = 0.0003t + C$. Taking
the natural logarithm of both sides gives $10000 - Q = c\,e^{-0.0003t}$. Since $Q(0) = 0$, the value of the constant is $c = 10000$. Hence the amount of chemical in the pond at any time
is $Q(t) = 10000(1 - e^{-0.0003t})$ *grams*. Note that 1 *year* $= 8760$ *hours*. Setting $t = 8760$, the amount of chemical present after *one year* is $Q(8760) = 9277.77$ *grams*, that is, 9.27777 *kilograms*.

(c). With the *accumulation* rate now equal to *zero*, the governing equation becomes $dQ/dt = -0.0003\,Q(t)$ *gm/hr*. Resetting the time variable, we now assign the new initial value as $Q(0) = 9277.77$ *grams*.

(d). The solution of the differential equation in Part (c) is $Q(t) = 9277.77\,e^{-0.0003t}$. Hence, one year *after* the source is removed, the amount of chemical in the pond is $Q(8760) = 670.1$ *grams*.

(e). Letting t be the amount of time after the source is removed, we obtain the equation $10 = 9277.77\, e^{-0.0003t}$. Taking the natural logarithm of both sides, $-\,0.0003\, t =$ $= ln(10/9277.77)$ or $t = 22,776$ *hours* $= 2.6$ *years* .

(f)



$15(a)$. It is assumed that dye is no longer entering the pool. In fact, the rate at which the dye *leaves* the pool is $200 \cdot [q(t)/60000]$ *kg/min* $= 200(60/1000)[q(t)/60]$ *gm per hour* .

Hence the equation that governs the amount of dye in the pool is

$$\frac{dq}{dt} = -\,0.2\, q \quad (gm/hr)\,.$$

The initial amount of dye in the pool is $q(0) = 5000$ *grams* .

(b). The solution of the governing differential equation, with the specified initial value, is $q(t) = 5000\, e^{-0.2\,t}$.

(c). The amount of dye in the pool after four hours is obtained by setting $t = 4$. That is, $q(4) = 5000\, e^{-0.8} = 2246.64$ *grams* . Since size of the pool is $60,000$ *gallons* , the *concentration* of the dye is 0.0374 *grams/gallon* .

(d). Let T be the time that it takes to reduce the concentration level of the dye to 0.02 *grams/gallon* . At that time, the amount of dye in the pool is $1,200$ *grams* . Using the answer in part (b), we have $5000\, e^{-0.2\,T} = 1200$. Taking the natural logarithm of both sides of the equation results in the required time $T = 7.14$ *hours* .

(e). Note that $0.2 = 200/1000$. Consider the differential equation

$$\frac{dq}{dt} = -\,\frac{r}{1000}\, q\,.$$

Here the parameter r corresponds to the *flow rate*, measured in *gallons per minute* . Using the same initial value, the solution is given by $q(t) = 5000\, e^{-r\,t/1000}$. In order to determine the appropriate flow rate, set $t = 4$ and $q = 1200$. (Recall that 1200 *gm* of dye has a concentration of 0.02 *gm/gal*). We obtain the equation $1200 = 5000\, e^{-r/250}$. Taking the natural logarithm of both sides of the equation results in the required flow rate $r = 357$ *gallons per minute*.

Section 1.3

1. The differential equation is *second order*, since the highest derivative in the equation is of order *two*. The equation is *linear*, since the left hand side is a linear function of y and
its derivatives.

3. The differential equation is *fourth order*, since the highest derivative of the function y is of order *four*. The equation is also *linear*, since the terms containing the dependent variable is linear in y and its derivatives.

4. The differential equation is *first order*, since the only derivative is of order *one*. The dependent variable is *squared*, hence the equation is *nonlinear*.

5. The differential equation is *second order*. Furthermore, the equation is *nonlinear*, since the dependent variable y is an argument of the *sine function*, which is *not* a linear function.

7. $y_1(t) = e^t \Rightarrow y_1'(t) = y_1''(t) = e^t$. Hence $y_1'' - y_1 = 0$.
Also, $y_2(t) = cosh\, t \Rightarrow y_1'(t) = sinh\, t$ and $y_2''(t) = cosh\, t$. Thus $y_2'' - y_2 = 0$.

9. $y(t) = 3t + t^2 \Rightarrow y'(t) = 3 + 2t$. Substituting into the differential equation, we have $t(3+2t) - (3t + t^2) = 3t + 2t^2 - 3t - t^2 = t^2$. Hence the given function is a solution.

10. $y_1(t) = t/3 \Rightarrow y_1'(t) = 1/3$ and $y_1''(t) = y_1'''(t) = y_1''''(t) = 0$. Clearly, $y_1(t)$ is a solution. Likewise, $y_2(t) = e^{-t} + t/3 \Rightarrow y_2'(t) = -e^{-t} + 1/3$, $y_2''(t) = e^{-t}$, $y_2'''(t) = -e^{-t}$, $y_2''''(t) = e^{-t}$. Substituting into the left hand side of the equation, we find that $e^{-t} + 4(-e^{-t}) + 3(e^{-t} + t/3) = e^{-t} - 4e^{-t} + 3e^{-t} + t = t$. Hence both functions are solutions of the differential equation.

11. $y_1(t) = t^{1/2} \Rightarrow y_1'(t) = t^{-1/2}/2$ and $y_1''(t) = -t^{-3/2}/4$. Substituting into the left hand side of the equation, we have

$$\begin{aligned} 2t^2\left(-t^{-3/2}/4\right) + 3t\left(t^{-1/2}/2\right) - t^{1/2} &= -t^{1/2}/2 + 3\,t^{1/2}/2 - t^{1/2} \\ &= 0 \end{aligned}$$

Likewise, $y_2(t) = t^{-1} \Rightarrow y_2'(t) = -t^{-2}$ and $y_2''(t) = 2\,t^{-3}$. Substituting into the left hand side of the differential equation, we have $2t^2(2\,t^{-3}) + 3t(-t^{-2}) - t^{-1} = 4\,t^{-1} -$ $-3\,t^{-1} - t^{-1} = 0$. Hence both functions are solutions of the differential equation.

12. $y_1(t) = t^{-2} \Rightarrow y_1'(t) = -2t^{-3}$ and $y_1''(t) = 6\,t^{-4}$. Substituting into the left hand side of the differential equation, we have $t^2(6\,t^{-4}) + 5t(-2t^{-3}) + 4\,t^{-2} = 6\,t^{-2} -$ $-10\,t^{-2} + 4\,t^{-2} = 0$. Likewise, $y_2(t) = t^{-2} ln\, t \Rightarrow y_2'(t) = t^{-3} - 2t^{-3} ln\, t$ and $y_2''(t) = -5\,t^{-4} + 6\,t^{-4} ln\, t$. Substituting into the left hand side of the equation, we have $t^2(-5\,t^{-4} + 6\,t^{-4} ln\, t) + 5t(t^{-3} - 2t^{-3} ln\, t) + 4(t^{-2} ln\, t) = -5\,t^{-2} + 6\,t^{-2} ln\, t +$

$+\,5\,t^{-2}-10\,t^{-2}ln\,t+4\,t^{-2}ln\,t=0$. Hence both functions are solutions of the differential equation.

13. $y(t)=(cos\,t)ln\,cos\,t+t\,sin\,t \Rightarrow y'(t)=-(sin\,t)ln\,cos\,t+t\,cos\,t$ and $y''(t)=-(cos\,t)ln\,cos\,t-t\,sin\,t+sec\,t$. Substituting into the left hand side of the differential equation, we have $(-(cos\,t)ln\,cos\,t-t\,sin\,t+sec\,t)+(cos\,t)ln\,cos\,t+$ $+\,t\,sin\,t=-(cos\,t)ln\,cos\,t-t\,sin\,t+sec\,t+(cos\,t)ln\,cos\,t+t\,sin\,t=sec\,t$. Hence the function $y(t)$ is a solution of the differential equation.

15. Let $y(t)=e^{rt}$. Then $y''(t)=r^2e^{rt}$, and substitution into the differential equation results in $r^2e^{rt}+2\,e^{rt}=0$. Since $e^{rt}\neq 0$, we obtain the algebraic equation $r^2+2=0$. The roots of this equation are $r_{1,2}=\pm\, i\sqrt{2}$.

17. $y(t)=e^{rt} \Rightarrow y'(t)=r\,e^{rt}$ and $y''(t)=r^2e^{rt}$. Substituting into the differential equation, we have $r^2e^{rt}+re^{rt}-6\,e^{rt}=0$. Since $e^{rt}\neq 0$, we obtain the algebraic equation $r^2+r-6=0$, that is, $(r-2)(r+3)=0$. The roots are $r_{1,2}=-3\,,2$.

18. Let $y(t)=e^{rt}$. Then $y'(t)=re^{rt}$, $y''(t)=r^2e^{rt}$ and $y'''(t)=r^3e^{rt}$. Substituting the derivatives into the differential equation, we have $r^3e^{rt}-3r^2e^{rt}+2re^{rt}=0$. Since $e^{rt}\neq 0$, we obtain the algebraic equation $r^3-3r^2+2r=0$. By inspection, it follows that $r(r-1)(r-2)=0$. Clearly, the roots are $r_1=0$, $r_2=1$ and $r_3=2$.

20. $y(t)=t^r \Rightarrow y'(t)=r\,t^{r-1}$ and $y''(t)=r(r-1)t^{r-2}$. Substituting the derivatives into the differential equation, we have $t^2[r(r-1)t^{r-2}]-4t(r\,t^{r-1})+4\,t^r=0$. After some algebra, it follows that $r(r-1)t^r-4r\,t^r+4\,t^r=0$. For $t\neq 0$, we obtain the algebraic equation $r^2-5r+4=0$. The roots of this equation are $r_1=1$ and $r_2=4$.

21. The order of the partial differential equation is *two*, since the highest derivative, in fact each one of the derivatives, is of *second order*. The equation is *linear*, since the left hand side is a linear function of the partial derivatives.

23. The partial differential equation is *fourth order*, since the highest derivative, and in fact each of the derivatives, is of order *four*. The equation is *linear*, since the left hand side is a linear function of the partial derivatives.

24. The partial differential equation is *second order*, since the highest derivative of the function $u(x,y)$ is of order *two*. The equation is *nonlinear*, due to the product $u\cdot u_x$ on the left hand side of the equation.

25. $u_1(x,y)=cos\,x\,cosh\,y \Rightarrow \frac{\partial^2 u_1}{\partial x^2}=-cos\,x\,cosh\,y$ and $\frac{\partial^2 u_1}{\partial y^2}=cos\,x\,cosh\,y$.
It is evident that $\frac{\partial^2 u_1}{\partial x^2}+\frac{\partial^2 u_1}{\partial y^2}=0$. Likewise, given $u_2(x,y)=ln(x^2+y^2)$, the second derivatives are

$$\frac{\partial^2 u_2}{\partial x^2} = \frac{2}{x^2+y^2} - \frac{4x^2}{(x^2+y^2)^2}$$
$$\frac{\partial^2 u_2}{\partial y^2} = \frac{2}{x^2+y^2} - \frac{4y^2}{(x^2+y^2)^2}$$

Adding the partial derivatives,

$$\begin{aligned}\frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2} &= \frac{2}{x^2+y^2} - \frac{4x^2}{(x^2+y^2)^2} + \frac{2}{x^2+y^2} - \frac{4y^2}{(x^2+y^2)^2} \\ &= \frac{4}{x^2+y^2} - \frac{4(x^2+y^2)}{(x^2+y^2)^2} \\ &= 0\,.\end{aligned}$$

Hence $u_2(x,y)$ is also a solution of the differential equation.

27. Let $u_1(x,t) = sin\,\lambda x\, sin\,\lambda at$. Then the second derivatives are

$$\frac{\partial^2 u_1}{\partial x^2} = -\lambda^2 sin\,\lambda x\, sin\,\lambda at$$
$$\frac{\partial^2 u_1}{\partial t^2} = -\lambda^2 a^2 sin\,\lambda x\, sin\,\lambda at$$

It is easy to see that $a^2\frac{\partial^2 u_1}{\partial x^2} = \frac{\partial^2 u_1}{\partial t^2}$. Likewise, given $u_2(x,t) = sin(x-at)$, we have

$$\frac{\partial^2 u_2}{\partial x^2} = -\,sin(x-at)$$
$$\frac{\partial^2 u_2}{\partial t^2} = -\,a^2 sin(x-at)$$

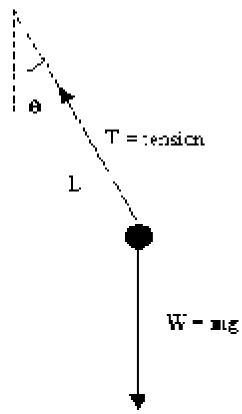
Clearly, $u_2(x,t)$ is also a solution of the partial differential equation.

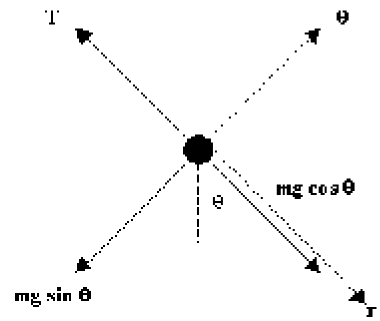
28. Given the function $u(x,t) = \sqrt{\pi/t}\; e^{-x^2/4\alpha^2 t}$, the partial derivatives are

$$u_{xx} = -\frac{\sqrt{\pi/t}\; e^{-x^2/4\alpha^2 t}}{2\alpha^2 t} + \frac{\sqrt{\pi/t}\; x^2 e^{-x^2/4\alpha^2 t}}{4\alpha^4 t^2}$$
$$u_t = -\frac{\sqrt{\pi t}\; e^{-x^2/4\alpha^2 t}}{2t^2} + \frac{\sqrt{\pi}\; x^2 e^{-x^2/4\alpha^2 t}}{4\alpha^2 t^2\sqrt{t}}$$

It follows that $\alpha^2\, u_{xx} = u_t = -\frac{\sqrt{\pi}\,\left(2\alpha^2 t - x^2\right) e^{-x^2/4\alpha^2 t}}{4\alpha^2 t^2\sqrt{t}}$.
Hence $u(x,t)$ is a solution of the partial differential equation.

29(a).





(b). The path of the particle is a circle, therefore *polar coordinates* are intrinsic to the problem. The variable r is radial distance and the angle θ is measured from the vertical. Newton's Second Law states that $\sum \mathbf{F} = m\mathbf{a}$. In the *tangential* direction, the equation of motion may be expressed as $\sum F_\theta = m\, a_\theta$, in which the *tangential acceleration*, that is, the linear acceleration *along* the path is $a_\theta = L\, d^2\theta/dt^2$. (a_θ is *positive* in the direction of increasing θ). Since the only force acting in the tangential direction is the component of weight, the equation of motion is

$$-\, mg\, sin\, \theta = mL\frac{d^2\theta}{dt^2}\, .$$

(Note that the equation of motion in the radial direction will include the tension in the rod).

(c). Rearranging the terms results in the differential equation

$$\frac{d^2\theta}{dt^2} + \frac{g}{L} sin\, \theta = 0\, .$$