Chapter Three

Section 3.1

1. Let $y = e^{rt}$, so that $y' = r\,e^{rt}$ and $y'' = r\,e^{rt}$. Direct substitution into the differential equation yields $(r^2 + 2r - 3)e^{rt} = 0$. Canceling the exponential, the characteristic equation is $r^2 + 2r - 3 = 0$. The roots of the equation are $r = -3\,,1$. Hence the general solution is $y = c_1e^t + c_2e^{-3t}$.

2. Let $y = e^{rt}$. Substitution of the assumed solution results in the characteristic equation $r^2 + 3r + 2 = 0$. The roots of the equation are $r = -2\,,-1$. Hence the general solution is $y = c_1e^{-t} + c_2e^{-2t}$.

4. Substitution of the assumed solution $y = e^{rt}$ results in the characteristic equation $2r^2 - 3r + 1 = 0$. The roots of the equation are $r = 1/2\,,1$. Hence the general solution is $y = c_1e^{t/2} + c_2e^t$.

6. The characteristic equation is $4r^2 - 9 = 0$, with roots $r = \pm 3/2$. Therefore the general solution is $y = c_1e^{-3t/2} + c_2e^{3t/2}$.

8. The characteristic equation is $r^2 - 2r - 2 = 0$, with roots $r = 1 \pm \sqrt{3}$. Hence the general solution is $y = c_1exp\left(1 - \sqrt{3}\right)t + c_2exp\left(1 + \sqrt{3}\right)t$.

9. Substitution of the assumed solution $y = e^{rt}$ results in the characteristic equation $r^2 + r - 2 = 0$. The roots of the equation are $r = -2\,,1$. Hence the general solution is $y = c_1e^{-2t} + c_2e^t$. Its derivative is $y' = -2c_1e^{-2t} + c_2e^t$. Based on the first condition, $y(0) = 1$, we require that $c_1 + c_2 = 1$. In order to satisfy $y'(0) = 1$, we find that $-2c_1 + c_2 = 1$. Solving for the constants, $c_1 = 0$ and $c_2 = 1$. Hence the specific solution is $y(t) = e^t$.

11. Substitution of the assumed solution $y = e^{rt}$ results in the characteristic equation $6r^2 - 5r + 1 = 0$. The roots of the equation are $r = 1/3\,,1/2$. Hence the general solution is $y = c_1e^{t/3} + c_2e^{t/2}$. Its derivative is $y' = c_1e^{t/3}/3 + c_2e^{t/2}/2$. Based on the first condition, $y(0) = 1$, we require that $c_1 + c_2 = 4$. In order to satisfy the condition $y'(0) = 1$, we find that $c_1/3 + c_2/2 = 0$. Solving for the constants, $c_1 = 12$ and $c_2 = -8$. Hence the specific solution is $y(t) = 12\,e^{t/3} - 8\,e^{t/2}$.

12. The characteristic equation is $r^2 + 3r = 0$, with roots $r = -3\,,0$. Therefore the general solution is $y = c_1 + c_2e^{-3t}$, with derivative $y' = -3\,c_2e^{-3t}$. In order to satisfy the initial conditions, we find that $c_1 + c_2 = -2$, and $-3\,c_2 = 3$. Hence the specific solution is $y(t) = -1 - e^{-3t}$.

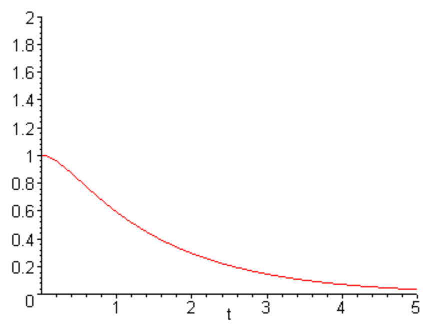
13. The characteristic equation is $r^2 + 5r + 3 = 0$, with roots

$$r_{1,2} = -\frac{5}{2} \pm \frac{\sqrt{13}}{2}\,.$$

The general solution is $y = c_1 exp\left(-5-\sqrt{13}\right)t/2 + c_2 exp\left(-5+\sqrt{13}\right)t/2$, with derivative

$$y' = \frac{-5-\sqrt{13}}{2}\,c_1 exp\left(-5-\sqrt{13}\right)t/2 + \frac{-5+\sqrt{13}}{2}\,c_2 exp\left(-5+\sqrt{13}\right)t/2\,.$$

In order to satisfy the initial conditions, we require that $c_1 + c_2 = 1$, and $\frac{-5-\sqrt{13}}{2}\,c_1 + \frac{-5+\sqrt{13}}{2}\,c_2 = 0$. Solving for the coefficients, $c_1 = \left(1 - 5/\sqrt{13}\right)/2$ and $c_2 = \left(1 + 5/\sqrt{13}\right)/2$.



14. The characteristic equation is $2r^2 + r - 4 = 0$, with roots

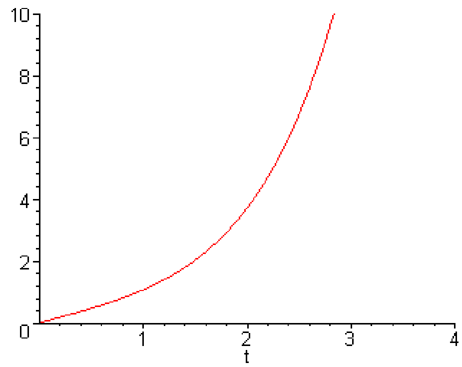
$$r_{1,2} = -\frac{1}{4} \pm \frac{\sqrt{33}}{4}\,.$$

The general solution is $y = c_1 exp\left(-1-\sqrt{33}\right)t/4 + c_2 exp\left(-1+\sqrt{33}\right)t/4$, with derivative

$$y' = \frac{-1-\sqrt{33}}{4}\,c_1 exp\left(-1-\sqrt{33}\right)t/4 + \frac{-1+\sqrt{33}}{4}\,c_2 exp\left(-1+\sqrt{33}\right)t/4\,.$$

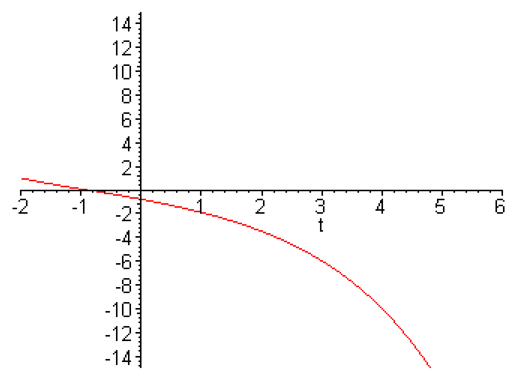
In order to satisfy the initial conditions, we require that $c_1 + c_2 = 0$, and $\frac{-1-\sqrt{33}}{4}\,c_1 + \frac{-1+\sqrt{33}}{4}\,c_2 = 1$. Solving for the coefficients, $c_1 = -2/\sqrt{33}$ and $c_2 = 2/\sqrt{33}$. The specific solution is

$$y(t) = -2\left[exp\left(-1-\sqrt{33}\right)t/4 - exp\left(-1+\sqrt{33}\right)t/4\right]/\sqrt{33}\,.$$



16. The characteristic equation is $4r^2 - 1 = 0$, with roots $r = \pm 1/2$. Therefore the general solution is $y = c_1 e^{-t/2} + c_2 e^{t/2}$. Since the initial conditions are specified at $t = -2$, is more convenient to write $y = d_1 e^{-(t+2)/2} + d_2 e^{(t+2)/2}$. The derivative is given by $y' = -\left[d_1 e^{-(t+2)/2}\right]/2 + \left[d_2 e^{(t+2)/2}\right]/2$. In order to satisfy the initial conditions, we find that $d_1 + d_2 = 1$, and $-d_1/2 + d_2/2 = -1$. Solving for the coefficients, $d_1 = 3/2$, and $d_2 = -1/2$. The specific solution is

$$\begin{aligned} y(t) &= \frac{3}{2} e^{-(t+2)/2} - \frac{1}{2} e^{(t+2)/2} \\ &= \frac{3}{2e} e^{-t/2} - \frac{e}{2} e^{t/2}. \end{aligned}$$



18. An algebraic equation with roots -2 and $-1/2$ is $2r^2 + 5r + 2 = 0$. This is the characteristic equation for the ODE $2y'' + 5y' + 2y = 0$.

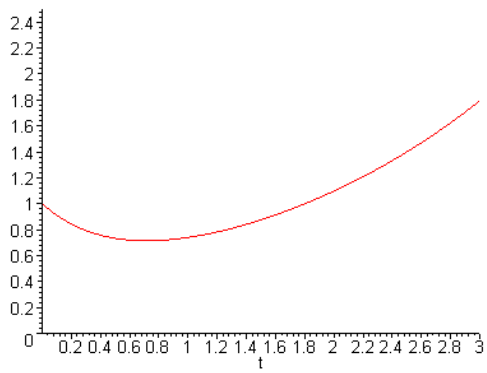
20. The characteristic equation is $2r^2 - 3r + 1 = 0$, with roots $r = 1/2, 1$. Therefore the general solution is $y = c_1 e^{t/2} + c_2 e^t$, with derivative $y' = c_1 e^{t/2}/2 + c_2 e^t$. In order to satisfy the initial conditions, we require $c_1 + c_2 = 2$ and $c_1/2 + c_2 = 1/2$. Solving for the coefficients, $c_1 = 3$, and $c_2 = -1$. The specific solution is $y(t) = 3e^{t/2} - e^t$. To find the *stationary point*, set $y' = 3e^{t/2}/2 - e^t = 0$. There is a unique solution, with $t_1 = ln(9/4)$. The maximum value is then $y(t_1) = 9/4$. To find

the *x-intercept*, solve the equation $3e^{t/2} - e^t = 0$. The solution is readily found to be $t_2 = ln9 \approx 2.1972$.

22. The characteristic equation is $4r^2 - 1 = 0$, with roots $r = \pm 1/2$. Hence the general solution is $y = c_1 e^{-t/2} + c_2 e^{t/2}$, with derivative $y' = -c_1 e^{-t/2}/2 + c_2 e^{t/2}/2$. Invoking the initial conditions, we require that $c_1 + c_2 = 2$ and $-c_1 + c_2 = \beta$. The specific solution is $y(t) = (1-\beta)e^{-t/2} + (1+\beta)e^{t/2}$. Based on the form of the solution, it is evident that as $t \to \infty$, $y(t) \to 0$ as long as $\beta = -1$.

23. The characteristic equation is $r^2 - (2\alpha - 1)r + \alpha(\alpha - 1) = 0$. Examining the coefficients, the roots are $r = \alpha$, $\alpha - 1$. Hence the general solution of the differential equation is $y(t) = c_1 e^{\alpha t} + c_2 e^{(\alpha-1)t}$. Assuming $\alpha \in \mathbb{R}$, all solutions will tend to *zero* as long as $\alpha < 0$. On the other hand, all solutions will become unbounded as long as $\alpha - 1 > 0$, that is, $\alpha > 1$.

25. $y(t) = 2\,e^{t/2}/5 + 3\,e^{-2t}/5$.



The minimum occurs at $(t_0, y_0) = (0.7167, 0.7155)$.

26(a). The characteristic roots are $r = -3, -2$. The solution of the initial value problem is $y(t) = (6+\beta)e^{-2t} - (4+\beta)e^{-3t}$.

(b). The maximum point has coordinates $t_0 = ln\left[\frac{3(4+\beta)}{2(6+\beta)}\right]$, $y_0 = \frac{4(6+\beta)^3}{27(4+\beta)^2}$.

(c). $y_0 = \frac{4(6+\beta)^3}{27(4+\beta)^2} \geq 4$, as long as $\beta \geq 6 + 6\sqrt{3}$.

(d). $\lim_{\beta\to\infty} t_0 = ln\,\frac{3}{2}$. $\lim_{\beta\to\infty} y_0 = \infty$.

29. Set $v = y'$ and $v' = y''$. Substitution into the ODE results in the first order equation $t\,v' + v = 1$. The equation is *linear*, and can be written as $(t\,v)' = 1$. Hence the general solution is $v = 1 + c_1/t$. Hence $y' = 1 + c_1/t$, and $y = t + c_1 ln\,t + c_2$.

31. Setting $v = y'$ and $v' = y''$, the transformed equation is $2t^2 v' + v^3 = 2t\,v$. This

is a *Bernoulli* equation, with $n = 3$. Let $w = v^{-2}$. Substitution of the new dependent variable yields $-t^2 w' + 1 = 2t\,w$, or $t^2 w' + 2t\,w = \ = 1$. Integrating, we find that $w = (t + c_1)/t^2$. Hence $v = \pm t/\sqrt{t + c_1}$, that is, $y' = \pm t/\sqrt{t + c_1}$. Integrating one more time results in $y(t) = \pm\frac{2}{3}(t - 2c_1)\sqrt{t + c_1} + c_2$. (Note that $v = 0$ is also a solution of the transformed equation).

32. Setting $v = y'$ and $v' = y''$, the transformed equation is $v' + v = e^{-t}$. This ODE is *linear*, with integrating factor $\mu(t) = e^t$. Hence $v = y' = (t + c_1)e^{-t}$. Integrating, we obtain $y(t) = -(t + c_1)e^{-t} + c_2$.

33. Set $v = y'$ and $v' = y''$. The resulting equation is $t^2 v' = v^2$. This equation is *separable*, with solution $v = y' = t/(1 + c_1 t)$. Integrating, the general solution is

$$y(t) = t/c_1 - c_1^{-2} ln|1 + c_1 t| + c_2\,,$$

as long as $c_1 \neq 0$. For $c_1 = 0$, the solution is $y(t) = t^2/2 + c_2$. Note that $v = 0$ is also a solution of the transformed equation.

35. Let $y' = v$ and $y'' = v\,dv/dy$. Then $v\,dv/dy + y = 0$ is the transformed equation for $v = v(y)$. This equation is *separable*, with $v\,dv = -y\,dy$. The solution is given by $v^2 = -y^2 + c_1$. Substituting for v , we find that $y' = \pm\sqrt{c_1 - y^2}$. This equation is *also*
separable, with solution $arcsin\big(y/\sqrt{c_1}\big) = \pm\,t + c_2$, or $y(t) = d_1 sin(t + d_2)$.

36. Let $y' = v$ and $y'' = v\,dv/dy$. It follows that $vdv/dy + yv^3 = 0$ is the differential equation for $v = v(y)$. This equation is *separable*, with $v^{-2}\,dv = -y\,dy$. The solution is given by $v = [y^2/2 + c_1]^{-1}$. Substituting for v , we find that $y' = [y^2/2 + c_1]^{-1}$. This equation is *also* separable, with $(y^2/2 + c_1)dy = dt$. The solution is defined *implicitly* by $y^3/6 + c_1 y + c_2 = t$.

38. Setting $y' = v$ and $y'' = vdv/dy$, the transformed equation is $y\,vdv/dy - v^3 = 0$. This equation is *separable*, with $v^{-2}\,dv = dy/y$. The solution is $v(y) = [c_1 - ln|y|]^{-1}$. Substituting for v , we obtain a *separable* equation, $(c_1 - ln|y|)dy = dx$. The solution is given *implicitly* by $c_2 y - y\,ln|y| + c_3 = t$.

39. Let $y' = v$ and $y'' = vdv/dy$. It follows that $vdv/dy + v^2 = 2e^{-y}$ is the equation for $v = v(y)$. Inspection of the left hand side suggests a substitution $w = v^2$. The resulting
equation is $dw/dy + 2w = 4e^{-y}$. This equation is *linear*, with integrating factor $\mu = e^{2y}$.
We obtain $d(e^{2y}\,w)/dy = 4\,e^{y}$, which upon integration yields $w(y) = 4\,e^{-y} + c_1 e^{-2y}$. Converting back to the original dependent variable, $y' = \pm e^{-y}\sqrt{4\,e^y + c_1}$. Separating variables, $e^y(4\,e^y + c_1)^{-1/2}dy = \pm\,dt$. Integration yields $\sqrt{4\,e^y + c_1} = \pm\,2t + c_2$.

41. Setting $y' = v$ and $y'' = vdv/dy$, the transformed equation is $vdv/dy - 3y^2 = 0$.

This equation is *separable*, with $vdv = 3y^2dy$. The solution is $y' = v = \sqrt{2y^3 + c_1}$. The *positive* root is chosen based on the initial conditions. Furthermore, when $t = 0$, $y = 2$, and $y' = v = 4$. The initial conditions require that $c_1 = 0$. It follows that $y' = \sqrt{2y^3}$. Separating variables and integrating, $1/\sqrt{y} = -t/\sqrt{2} + c_2$. Hence the solution is $y(t) = 2/(1-t)^2$.

42. Setting $v = y'$ and $v' = y''$, the transformed equation is $(1+t^2)v' + 2t\,v =$ $= -3t^{-2}$. Rewrite the equation as $v' + 2t\,v/(1+t^2) = -3t^{-2}/(1+t^2)$. This equation is *linear*, with integrating factor $\mu = 1 + t^2$. Hence we have

$$\left[(1+t^2)v\right]' = -3t^{-2}.$$

Integrating both sides, $v = 3t^{-1}/(1+t^2) + c_1/(1+t^2)$. Invoking the initial condition $v(1) = -1$, we require that $c_1 = -5$. Hence $y' = (3-5t)/(t+t^3)$. Integrating, we obtain $y(t) = \frac{3}{2}ln[t^2/(1+t^2)] - 5\,arctan(t) + c_2$. Based on the initial condition $y(1) = 2$, we find that $c_2 = \frac{3}{2}ln\,2 + \frac{5}{4}\pi + 2$.

Section 3.2

1.

$$W\left(e^{2t},e^{-3t/2}\right)=\begin{vmatrix} e^{2t} & e^{-3t/2} \\ 2e^{2t} & -\frac{3}{2}e^{-3t/2} \end{vmatrix} = -\frac{7}{2}e^{t/2}.$$

3.

$$W\left(e^{-2t},t\,e^{-2t}\right)=\begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & (1-2t)e^{-2t} \end{vmatrix} = e^{-4t}\,.$$

5.

$$W\left(e^{t}sin\,t\,,e^{t}cos\,t\right)=\begin{vmatrix} e^{t}sin\,t & e^{t}cos\,t \\ e^{t}(sin\,t+cos\,t) & e^{t}(cos\,t-sin\,t) \end{vmatrix} = -\,e^{2t}\,.$$

6.

$$W\left(cos^2\theta\,,1+cos\,2\theta\right)=\begin{vmatrix} cos^2\theta & 1+cos\,2\theta \\ -\,2sin\,\theta\,cos\,\theta & -\,2\,sin\,2\theta \end{vmatrix} = 0\,.$$

7. Write the equation as $y''+(3/t)y'=1$. $p(t)=3/t$ is continuous for all $t>0$. Since $t_0>0$, the IVP has a unique solution for all $t>0$.

9. Write the equation as $y''+\frac{3}{t-4}y'+\frac{4}{t(t-4)}y=\frac{2}{t(t-4)}$. The coefficients are not continuous at $t=0$ and $t=4$. Since $t_0\in(0\,,4)$, the largest interval is $0<t<4$.

10. The coefficient $3ln|t|$ is discontinuous at $t=0$. Since $t_0>0$, the largest interval of existence is $0<t<\infty$.

11. Write the equation as $y''+\frac{x}{x-3}y'+\frac{ln|x|}{x-3}y=0$. The coefficients are discontinuous at $x=0$ and $x=3$. Since $x_0\in(0\,,3)$, the largest interval is $0<x<3$.

13. $y_1''=2$. We see that $t^2(2)-2(t^2)=0$. $y_2''=2\,t^{-3}$, with $t^2(y_2'')-2(y_2)=0$. Let $y_3=c_1t^2+c_2t^{-1}$, then $y_3''=2c_1+2c_2t^{-3}$. It is evident that y_3 is also a solution.

16. No. Substituting $y=sin(t^2)$ into the differential equation,

$$-\,4t^2sin\left(t^2\right)+2cos\left(t^2\right)+2t\,cos\left(t^2\right)p(t)+sin\left(t^2\right)q(t)=0\,.$$

For the equation to be valid, we must have $p(t)=\,-\,1/t$, which is *not* continuous, or even defined, at $t=0$.

17. $W(e^{2t}, g(t)) = e^{2t}g'(t) - 2e^{2t}g(t) = 3e^{4t}$. Dividing both sides by e^{2t}, we find that g must satisfy the ODE $g' - 2g = 3e^{2t}$. Hence $g(t) = 3t\,e^{2t} + c\,e^{2t}$.

19. $W(f\,,g) = fg' - f'g$. Also, $W(u\,,v) = W(2f - g\,, f + 2g)$. Upon evaluation, $W(u\,,v) = 5fg' - 5f'g = 5W(f\,,g)$.

20. $W(f\,,g) = fg' - f'g = t\,cos\,t - sin\,t$, and $W(u\,,v) = -4fg' + 4f'g$. Hence $W(u\,,v) = -4t\,cos\,t + 4sin\,t$.

22. The general solution is $y = c_1e^{-3t} + c_2e^{-t}$. $W(e^{-3t}, e^{-t}) = 2e^{-4t}$, and hence the exponentials form a *fundamental set* of solutions. On the other hand, the *fundamental solutions* must also satisfy the conditions $y_1(1) = 1\,, y_1'(1) = 0\,; y_2(1) = 0\,, y_2'(1) = 1$. For y_1, the initial conditions require $c_1 + c_2 = e\,,\ -3c_1 - c_2 = 0$. The coefficients are $c_1 = -e^3/2\,, c_2 = 3e/2$. For the solution, y_2, the initial conditions require $c_1 + c_2 = 0$, $-3c_1 - c_2 = e$. The coefficients are $c_1 = -e^3/2\,, c_2 = e/2$. Hence the fundamental solutions are $\{y_1 = -\frac{1}{2}e^{-3(t-1)} + \frac{3}{2}e^{-(t-1)},\ y_2 = -\frac{1}{2}e^{-3(t-1)} + \frac{1}{2}e^{-(t-1)}\}$.

23. Yes. $y_1'' = -4\,cos\,2t\,;\ y_2'' = -4\,sin\,2t$. $W(cos\,2t\,, sin\,2t) = 2$.

24. Clearly, $y_1 = e^t$ is a solution. $y_2' = (1+t)e^t, y_2'' = (2+t)e^t$. Substitution into the ODE results in $(2+t)e^t - 2(1+t)e^t + t\,e^t = 0$. Furthermore, $W(e^t, te^t) = e^{2t}$. Hence the solutions form a fundamental set of solutions.

26. Clearly, $y_1 = x$ is a solution. $y_2' = cos\,x\,, y_2'' = -sin\,x$. Substitution into the ODE results in $(1 - x\,cot\,x)(-sin\,x) - x(cos\,x) + sin\,x = 0$. $W(y_1, y_2) = x\,cos$
$x - sin\,x$,
which is *nonzero* for $0 < x < \pi$. Hence $\{x\,, sin\,x\}$ is a fundamental set of solutions.

28. $P = 1\,, Q = x\,, R = 1$. We have $P'' - Q' + R = 0$. The equation is *exact*. Note that $(y')' + (xy)' = 0$. Hence $y' + xy = c_1$. This equation is *linear*, with integrating factor $\mu = e^{x^2/2}$. Therefore the general solution is

$$y(x) = c_1exp\big(-x^2/2\big)\int_{x_0}^{x} exp\big(u^2/2\big)du + c_2exp\big(-x^2/2\big).$$

29. $P = 1\,, Q = 3x^2\,, R = x$. Note that $P'' - Q' + R = -5x$, and therefore the differential equation is *not* exact.

31. $P = x^2\,, Q = x\,, R = -1$. We have $P'' - Q' + R = 0$. The equation is *exact*. Write the equation as $(x^2y')' - (xy)' = 0$. Integrating, we find that $x^2y' - x\,y = c$. Divide both sides of the ODE by x^2. The resulting equation is *linear*, with integrating factor $\mu = 1/x$. Hence $(y/x)' = c\,x^{-3}$. The solution is $y(t) = c_1x^{-1} + c_2x$.

33. $P = x^2\,, Q = x\,, R = x^2 - \nu^2$. Hence the coefficients are $2P' - Q = 3x$ and $P'' - Q' + R = x^2{+}1 - \nu^2$. The *adjoint* of the original differential equation is given by $x^2\mu'' + 3x\,\mu'{+}(x^2{+}1 - \nu^2)\mu = 0\,.$

35. $P = 1\,, Q = 0\,, R = \,-\,x\,.$ Hence the coefficients are given by $2P' - Q = 0$ and $P'' - Q' + R = \,-\,x\,.$ Therefore the *adjoint* of the original equation is $\mu'' - x\,\mu = 0\,.$

Section 3.3

1. Suppose that $\alpha f(t) + \beta g(t) = 0$, that is, $\alpha(t^2 + 5t) + \beta(t^2 - 5t) = 0$ on some interval I. Then $(\alpha + \beta)t^2 + 5(\alpha - \beta)t = 0, \forall t \in I$. Since a quadratic .has at most *two*
roots, we must have $\alpha + \beta = 0$ and $\alpha - \beta = 0$. The *only* solution is $\alpha = \beta = 0$. Hence the two functions are linearly *independent*.

3. Suppose that $e^{\lambda t} cos\, \mu t = A\, e^{\lambda t} sin\, \mu t$, for some $A \neq 0$, on an interval I. Since the function $sin\, \mu t \neq 0$ on some *subinterval* $I_0 \subset I$, we conclude that $tan\, \mu t = A$ on I_0. This is clearly a contradiction, hence the functions are linearly *independent*.

4. Obviously, $f(x) = e\, g(x)$ for all real numbers x. Hence the functions are linearly *dependent*.

5. Here $f(x) = 3g(x)$ for all real numbers. Hence the functions are linearly *dependent*.

8. Note that $f(x) = g(x)$ for $x \in [0, \infty)$, and $f(x) = -g(x)$ for $x \in (-\infty, 0]$. It follows that the functions are linearly *dependent* on $\mathbb{R}^+$ and $\mathbb{R}^-$. Nevertheless, they are linearly *independent* on *any* open interval containing *zero*.

9. Since $W(t) = t\, sin^2 t$ has only *isolated* zeros, $W(t)$ cannot identically vanish on any open interval. Hence the functions are linearly *independent*.

10. Same argument as in Prob. 9.

11. By linearity of the differential operator, $c_1 y_1$ and $c_2 y_2$ are also solutions. Calculating
the Wronskian, $W(c_1 y_1, c_2 y_2) = (c_1 y_1)(c_2 y_2)' - (c_1 y_1)'(c_2 y_2) = c_1 c_2\, W(y_1, y_2)$. Since $W(y_1, y_2)$ is not *identically zero*, neither is $W(c_1 y_1, c_2 y_2)$.

13. Direct calculation results in

$$\begin{aligned} W(a_1 y_1 + a_2 y_2, b_1 y_1 + b_2 y_2) &= a_1 b_2 W(y_1, y_2) - b_1 a_2 W(y_1, y_2) \\ &= (a_1 b_2 - a_2 b_1) W(y_1, y_2). \end{aligned}$$

Hence the combinations are also linearly independent as long as $a_1 b_2 - a_2 b_1 \neq 0$.

14. Let $\alpha(\mathbf{i} + \mathbf{j}) + \beta(\mathbf{i} - \mathbf{j}) = 0\,\mathbf{i} + 0\,\mathbf{j}$. Then $\alpha + \beta = 0$ and $\alpha - \beta = 0$. The only solution is $\alpha = \beta = 0$. Hence the given vectors are linearly independent. Furthermore, any vector $a_1 \mathbf{i} + a_2 \mathbf{j} = \left(\frac{a_1}{2} + \frac{a_2}{2}\right)(\mathbf{i} + \mathbf{j}) + \left(\frac{a_1}{2} - \frac{a_2}{2}\right)(\mathbf{i} - \mathbf{j})$.

16. Writing the equation in standard form, we find that $P(t) = sin\, t / cos\, t$. Hence the Wronskian is $W(t) = b\, exp\left(-\int \frac{sin\, t}{cos\, t} dt\right) = b\, exp(ln|cos\, t|) = b\, cos\, t$, in which b is some constant.

17. After writing the equation in standard form, we have $P(x) = 1/x$. The Wronskian is $W(t) = c\,exp\big(-\int \frac{1}{x}dx\big) = c\,exp(-ln|x|) = c/|x|$, in which c is some constant.

18. Writing the equation in standard form, we find that $P(x) = -2x/(1-x^2)$. The Wronskian is $W(t) = c\,exp\big(-\int \frac{-2x}{1-x^2}dx\big) = c\,exp(-ln|1-x^2|) = c|1-x^2|^{-1}$, in which c is some constant.

19. Rewrite the equation as $p(t)y'' + p'(t)y' + q(t)y = 0$. After writing the equation in standard form, we have $P(t) = p'(t)/p(t)$. Hence the Wronskian is

$$W(t) = c\,exp\bigg(-\int \frac{p'(t)}{p(t)}dt\bigg) = c\,exp(-ln\,p(t)) = c/p(t)\,.$$

21. The Wronskian associated with the solutions of the differential equation is given by $W(t) = c\,exp\big(-\int \frac{-2}{t^2}dt\big) = c\,exp(-2/t)$. Since $W(2) = 3$, it follows that for the hypothesized set of solutions, $c = 3\,e$. Hence $W(4) = 3\sqrt{e}$.

22. For the given differential equation, the Wronskian satisfies the first order differential equation $W' + p(t)W = 0$. Given that W is *constant*, it is necessary that $p(t) \equiv 0$.

23. Direct calculation shows that

$$\begin{aligned} W(f\,g\,,f\,h) &= (fg)(fh)' - (fg)'(fh) \\ &= (fg)(f'h + fh') - (f'g + fg')(fh) \\ &= f^2\,W(g\,,h)\,. \end{aligned}$$

25. Since y_1 and y_2 are solutions, they are differentiable. The hypothesis can thus be restated as $y_1'(t_0) = y_2'(t_0) = 0$ at some point t_0 in the interval of definition. This implies that $W(y_1\,,y_2)(t_0) = 0$. But $W(y_1\,,y_2)(t_0) = c\,exp\big(-\int p(t)dt\big)$, which *cannot* be equal to zero, unless $c = 0$. Hence $W(y_1\,,y_2) \equiv 0$, which is ruled out for a fundamental set of solutions.

Section 3.4

2. $exp(2-3i) = e^2 e^{-3i} = e^2(cos\,3 - i\,sin\,3)$.

3. $e^{i\pi} = cos\,\pi + i\,sin\,\pi = -1$.

4. $exp\big(2-\frac{\pi}{2}i\big) = e^2\big(cos\,\frac{\pi}{2} - i\,sin\,\frac{\pi}{2}\big) = -e^2\,i$.

6. $\pi^{-1+2i} = exp[(-1+2i)ln\,\pi] = exp(-ln\,\pi)exp(2\,ln\,\pi\,i) = \frac{1}{\pi}\,exp(2\,ln\,\pi\,i) =$
$= \frac{1}{\pi}[cos\,(2\,ln\,\pi) + i\,sin\,(2\,ln\,\pi)]$.

8. The characteristic equation is $r^2 - 2r + 6 = 0$, with roots $r = 1 \pm i\sqrt{5}$. Hence the general solution is $y = c_1 e^t cos\,\sqrt{5}\,t + c_2\,e^t sin\,\sqrt{5}\,t$.

9. The characteristic equation is $r^2 + 2r - 8 = 0$, with roots $r = -4, 2$. The roots are *real* and different, hence the general solution is $y = c_1 e^{-4t} + c_2\,e^{2t}$.

10. The characteristic equation is $r^2 + 2r + 2 = 0$, with roots $r = -1 \pm i$. Hence the general solution is $y = c_1 e^{-t} cos\,t + c_2\,e^{-t} sin\,t$.

12. The characteristic equation is $4r^2 + 9 = 0$, with roots $r = \pm\frac{3}{2}\,i$. Hence the general solution is $y = c_1 cos\,\frac{3}{2}t + c_2\,sin\,\frac{3}{2}t$.

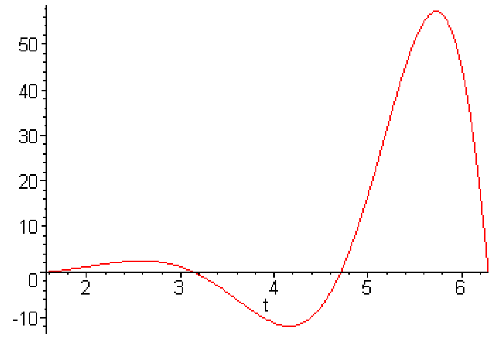
13. The characteristic equation is $r^2 + 2r + 1.25 = 0$, with roots $r = -1 \pm \frac{1}{2}i$. Hence the general solution is $y = c_1 e^{-t} cos\,\frac{1}{2}t + c_2\,e^{-t} sin\,\frac{1}{2}t$.

15. The characteristic equation is $r^2 + r + 1.25 = 0$, with roots $r = -\frac{1}{2} \pm i$. Hence the general solution is $y = c_1 e^{-t/2} cos\,t + c_2\,e^{-t/2} sin\,t$.

16. The characteristic equation is $r^2 + 4r + 6.25 = 0$, with roots $r = -2 \pm \frac{3}{2}\,i$. Hence the general solution is $y = c_1 e^{-2t} cos\,\frac{3}{2}t + c_2\,e^{-2t} sin\,\frac{3}{2}t$.

17. The characteristic equation is $r^2 + 4 = 0$, with roots $r = \pm\,2i$. Hence the general solution is $y = c_1 cos\,2t + c_2\,sin\,2t$. Its derivative is $y' = -2c_1 sin\,2t + 2c_2\,cos\,2t$. Based on the first condition, $y(0) = 0$, we require that $c_1 = 0$. In order to satisfy the condition $y'(0) = 1$, we find that $2c_2 = 1$. The constants are $c_1 = 0$ and $c_2 = 1/2$. Hence the specific solution is $y(t) = \frac{1}{2} sin\,2t$.

19. The characteristic equation is $r^2 - 2r + 5 = 0$, with roots $r = 1 \pm 2i$. Hence the general solution is $y = c_1 e^t cos\,2t + c_2\,e^t sin\,2t$. Based on the condition, $y(\pi/2) = 0$, we require that $c_1 = 0$. It follows that $y = c_2\,e^t sin\,2t$, and so the first derivative is $y' = c_2\,e^t sin\,2t + 2c_2\,e^t cos\,2t$. In order to satisfy the condition $y'(\pi/2) = 2$, we find that $-2e^{\pi/2}c_2 = 2$. Hence we have $c_2 = -e^{-\pi/2}$. Therefore the specific solution is $y(t) = -e^{t-\pi/2}\,sin\,2t$.

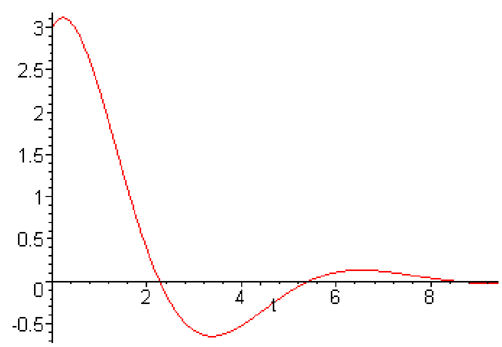


20. The characteristic equation is $r^2 + 1 = 0$, with roots $r = \pm i$. Hence the general solution is $y = c_1 cos\, t + c_2\, sin\, t$. Its derivative is $y' = -c_1 sin\, t + c_2\, cos\, t$. Based on the first condition, $y(\pi/3) = 2$, we require that $c_1 + \sqrt{3}\, c_2 = 4$. In order to satisfy the condition $y'(\pi/3) = -4$, we find that $-\sqrt{3}\, c_1 + c_2 = -8$. Solving these for the constants, $c_1 = 1 + 2\sqrt{3}$ and $c_2 = \sqrt{3} - 2$. Hence the specific solution is a steady oscillation, given by $y(t) = \left(1 + 2\sqrt{3}\right) cos\, t + \left(\sqrt{3} - 2\right) sin\, t$.

21. From Prob. 15, the general solution is $y = c_1 e^{-t/2} cos\, t + c_2\, e^{-t/2} sin\, t$. Invoking the first initial condition, $y(0) = 3$, which implies that $c_1 = 3$. Substituting, it follows that $y = 3e^{-t/2} cos\, t + c_2\, e^{-t/2} sin\, t$, and so the first derivative is

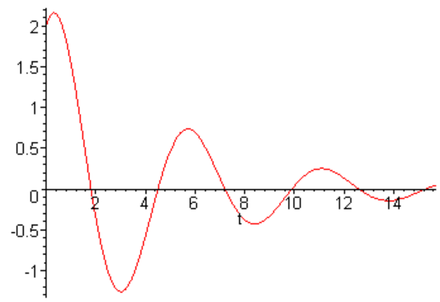
$$y' = -\frac{3}{2} e^{-t/2} cos\, t - 3e^{-t/2} sin\, t + c_2\, e^{-t/2} cos\, t - \frac{c_2}{2} e^{-t/2} sin\, t\,.$$

Invoking the initial condition, $y'(0) = 1$, we find that $-\frac{3}{2} + c_2 = 1$, and so $c_2 = \frac{5}{2}$. Hence the specific solution is $y(t) = 3e^{-t/2} cos\, t + \frac{5}{3}\, e^{-t/2} sin\, t$.



24(a). The characteristic equation is $5r^2 + 2r + 7 = 0$, with roots $r = -\frac{1}{5} \pm i \frac{\sqrt{34}}{5}$. The solution is $u = c_1 e^{-t/5} cos\, \frac{\sqrt{34}}{5} t + c_2 e^{-t/5} sin\, \frac{\sqrt{34}}{5} t$. Invoking the given initial conditions, we obtain the equations for the coefficients: $c_1 = 2$, $-2 + \sqrt{34}\, c_2 = 5$. That is, $c_1 = 2$, $c_2 = 7/\sqrt{34}$. Hence the specific solution is

$$u(t) = 2e^{-t/5}cos\,\frac{\sqrt{34}}{5}t + \frac{7}{\sqrt{34}}e^{-t/5}sin\,\frac{\sqrt{34}}{5}t\,.$$



(b). Based on the graph of $u(t), T$ is in the interval $14 < t < 16$. A numerical solution on that interval yields $T \approx 14.5115$.

26(a). The characteristic equation is $r^2 + 2a\,r + (a^2+1) = 0$, with roots $r = \,-\,a \pm i$. Hence the general solution is $y(t) = c_1e^{-at}cos\,t + c_2e^{-at}sin\,t$. Based on the initial conditions, we find that $c_1 = 1$ and $c_2 = a$. Therefore the specific solution is given by

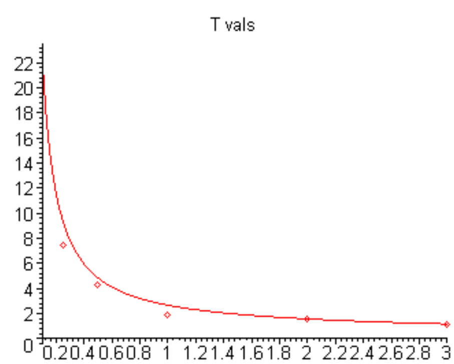
$$\begin{aligned} y(t) &= e^{-at}cos\,t + a\,e^{-at}sin\,t \\ &= \sqrt{1+a^2}\;e^{-at}cos\,(t-\phi), \end{aligned}$$

in which $\phi = tan^{-1}(a)$.

(b). For estimation, note that $|y(t)| \leq \sqrt{1+a^2}\;e^{-at}$. Now consider the inequality $\sqrt{1+a^2}\;e^{-at} \leq 1/10$. The inequality holds for $t \geq \frac{1}{a}ln\left[10\sqrt{1+a^2}\right]$. Therefore $T \leq \frac{1}{a}ln\left[10\sqrt{1+a^2}\right]$. Setting $a = 1$, numerical analysis gives $T \approx 1.8763$.

(c). Similarly, $T_{1/4} \approx 7.4284\,, T_{1/2} \approx 4.3003\,, T_2 \approx 1.5116\,, T_3 \approx 1.1496$.

(d).



Note that the estimates T_a approach the graph of $\frac{1}{a}ln\left[10\sqrt{1+a^2}\right]$ as a gets large.

27. Direct calculation gives the result. On the other hand, it was shown in Prob. 3.3.23 that $W(f\,g\,,f\,h)=f^2W(g\,,h)$. Hence

$$\begin{aligned} W\left(e^{\lambda t}cos\,\mu t\,,e^{\lambda t}sin\,\mu t\right) &= e^{2\lambda t}\,W(cos\,\mu t\,,sin\,\mu t) \\ &= e^{2\lambda t}\left[cos\,\mu t(sin\,\mu t)'-(cos\,\mu t)'sin\,\mu t\right] \\ &= \mu\,e^{2\lambda t}. \end{aligned}$$

28(a). Clearly, y_1 and y_2 are solutions. Also, $W(cos\,t, sin\,t)=cos^2t+sin^2t=1$.

(b). $y'=i\,e^{it}\,,y''=i^2\,e^{it}=\,-\,e^{it}$. Evidently, y is a solution and so $y=c_1y_1+c_2y_2$.

(c). Setting $t=0\,,\ 1=c_1cos\,0+c_2\,sin\,0$, and $c_1=0$. Differentiating, $i\,e^{it}=c_2\,cos\,t$. Setting $t=0\,,\ i=c_2\,cos\,0$ and hence $c_2=i$. Therefore $e^{it}=cos\,t+i\,sin\,t$.

29. *Euler's formula* is $e^{it}=cos\,t+i\,sin\,t$. It follows that $e^{-it}=cos\,t-i\,sin\,t$. Adding these equation, $e^{it}+e^{-it}=2\,cos\,t$. Subtracting the two equations results in $e^{it}-e^{-it}=2i\,sin\,t$.

30. Let $r_1=\lambda_1+i\mu_1$, and $r_2=\lambda_2+i\mu_2$. Then

$$\begin{aligned} exp(r_1+r_2)t &= exp[(\lambda_1+\lambda_2)t+i(\mu_1+\mu_2)t] \\ &= e^{(\lambda_1+\lambda_2)t}[cos(\mu_1+\mu_2)t+i\,sin(\mu_1+\mu_2)t] \\ &= e^{(\lambda_1+\lambda_2)t}[(cos\,\mu_1 t+\ isin\,\mu_1 t)(cos\,\mu_2 t+\ isin\,\mu_2 t)] \\ &= e^{\lambda_1 t}(cos\,\mu_1 t+\ isin\,\mu_1 t)\cdot e^{\lambda_2 t}(cos\,\mu_1 t+\ isin\,\mu_1 t) \end{aligned}$$

Hence $e^{(r_1+r_2)t}=e^{r_1t}\,e^{r_2t}$.

32. If $\phi(t)=u(t)+i\,v(t)$ is a solution, then

$$(u+iv)''+p(t)(u+iv)'+q(t)(u+iv)=0\,,$$

and $(u''+iv'')+p(t)(u'+iv')+q(t)(u+iv)=0$. After expanding the equation and separating the *real* and *imaginary* parts,

$$\begin{aligned} u''+p(t)u'+q(t)u=0 \\ v''+p(t)v'+q(t)v=0 \end{aligned}$$

Hence both $u(t)$ and $v(t)$ are solutions.

34(a). By the *chain rule*, $y(x)'=\frac{dy}{dx}\,x'$. In general, $\frac{dz}{dt}=\frac{dz}{dx}\,\frac{dx}{dt}$. Setting $z=\frac{dy}{dt}$, we have $\frac{d^2y}{dt^2}=\frac{dz}{dx}\,\frac{dx}{dt}=\frac{d}{dx}\left[\frac{dy}{dx}\,\frac{dx}{dt}\right]\frac{dx}{dt}=\left[\frac{d^2y}{dx^2}\,\frac{dx}{dt}\right]\frac{dx}{dt}+\frac{dy}{dx}\,\frac{d}{dx}\left[\frac{dx}{dt}\right]\frac{dx}{dt}$. However, $\frac{d}{dx}\left[\frac{dx}{dt}\right]\frac{dx}{dt}=\left[\frac{d^2x}{dt^2}\right]\frac{dt}{dx}\cdot\frac{dx}{dt}=\frac{d^2x}{dt^2}$. Hence $\frac{d^2y}{dt^2}=\frac{d^2y}{dx^2}\left[\frac{dx}{dt}\right]^2+\frac{dy}{dx}\,\frac{d^2x}{dt^2}$.

(b). Substituting the results in Part(a) into the general ODE, $y'' + p(t)y' + q(t)y = 0$, `we find that

$$\frac{d^2y}{dx^2}\left[\frac{dx}{dt}\right]^2 + \frac{dy}{dx}\frac{d^2x}{dt^2} + p(t)\frac{dy}{dx}\frac{dx}{dt} + q(t)y = 0\,.$$

Collecting the terms,

$$\left[\frac{dx}{dt}\right]^2\frac{d^2y}{dx^2} + \left[\frac{d^2x}{dt^2} + p(t)\frac{dx}{dt}\right]\frac{dy}{dx} + q(t)y = 0\,.$$

(c). Assuming $\left[\frac{dx}{dt}\right]^2 = k\,q(t)$, and $q(t) > 0$, we find that $\frac{dx}{dt} = \sqrt{k\,q(t)}$, which can be integrated. That is, $x = \xi(t) = \int \sqrt{k\,q(t)}\;dt$.

(d). Let $k = 1$. It follows that $\frac{d^2x}{dt^2} + p(t)\frac{dx}{dt} = \frac{d\xi}{dt} + p(t)\xi(t) = \frac{q'}{2\sqrt{q}} + p\,\sqrt{q}$. Hence

$$\left[\frac{d^2x}{dt^2} + p(t)\frac{dx}{dt}\right] \Big/ \left[\frac{dx}{dt}\right]^2 = \frac{q'(t) + 2p(t)q(t)}{2[q(t)]^{3/2}}\,.$$

As long as $dx/dt \neq 0$, the differential equation can be expressed as

$$\frac{d^2y}{dx^2} + \left[\frac{q'(t) + 2p(t)q(t)}{2[q(t)]^{3/2}}\right]\frac{dy}{dx} + y = 0\,.$$

* For the case $q(t) < 0$, write $q(t) = -[-q(t)]$, and set $\left[\frac{dx}{dt}\right]^2 = -q(t)$.

36. $p(t) = 3t$ and $q(t) = t^2$. We have $x = \int t\,dt = t^2/2$. Furthermore,

$$\frac{q'(t) + 2p(t)q(t)}{2[q(t)]^{3/2}} = \left(1 + 3t^2\right)/t^2\,.$$

The ratio is *not* constant, and therefore the equation cannot be transformed.

37. $p(t) = t - 1/t$ and $q(t) = t^2$. We have $x = \int t\,dt = t^2/2$. Furthermore,

$$\frac{q'(t) + 2p(t)q(t)}{2[q(t)]^{3/2}} = 1\,.$$

The ratio is constant, and therefore the equation can be transformed. From Prob. 35, the transformed equation is

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0\,.$$

Based on the methods in this section, the characteristic equation is $r^2 + r + 1 = 0$, with roots $r = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$. The general solution is

$$y(x) = c_1 e^{-x/2} cos\,\sqrt{3}\,x/2 + c_2\,e^{-x/2} sin\,\sqrt{3}\,x/2\,.$$

Since $x = t^2/2$, the solution in the original variable t is

$$y(t) = e^{-t^2/4}\Big[c_1 cos\,\big(\sqrt{3}\;t^2/4\big) + c_2\,sin\,\big(\sqrt{3}\;t^2/4\big)\Big]\,.$$

40. $p(t) = 4/t$ and $q(t) = 2/t^2$. We have $x = \sqrt{2}\int t^{-1}dt = \sqrt{2}\,ln\,t$. Furthermore,

$$\frac{q'(t) + 2p(t)q(t)}{2[q(t)]^{3/2}} = \frac{3}{\sqrt{2}}\,.$$

The ratio is constant, and therefore the equation can be transformed. In fact, we obtain

$$\frac{d^2y}{dx^2} + \frac{3}{\sqrt{2}}\frac{dy}{dx} + y = 0\,.$$

Based on the methods in this section, the characteristic equation is $\sqrt{2}\;r^2 + 3r + \sqrt{2} = 0$, with roots $r = \,-\sqrt{2}\,,\;-1/\sqrt{2}$. The general solution is

$$y(x) = c_1 e^{-\sqrt{2}\,x} + c_2\,e^{-x/\sqrt{2}}.$$

Since $x = \sqrt{2}\;ln\,t$, the solution in the original variable t is

$$\begin{aligned} y(t) &= c_1 e^{-2\,ln\,t} + c_2\,e^{-ln\,t} \\ &= c_1 t^{-2} + c_2\,t^{-1}. \end{aligned}$$

41. $p(t) = 3/t$ and $q(t) = 1.25/t^2$. We have $x = \sqrt{1.25}\int t^{-1}dt = \sqrt{1.25}\,ln\,t$. Checking the feasibility of the transformation,

$$\frac{q'(t) + 2p(t)q(t)}{2[q(t)]^{3/2}} = \frac{4}{\sqrt{5}}\,.$$

The ratio is constant, and therefore the equation can be transformed. In fact, we obtain

$$\frac{d^2y}{dx^2} + \frac{4}{\sqrt{5}}\frac{dy}{dx} + y = 0\,.$$

Based on the methods in this section, the characteristic equation is $\sqrt{5}\,r^2 + 4r + \sqrt{5} = 0$, with roots $r = \,-\frac{2}{\sqrt{5}} \pm i\frac{1}{\sqrt{5}}$. The general solution is

$$y(x) = c_1 e^{-2x/\sqrt{5}} cos\,x/\sqrt{5} + c_2\,e^{-2x/\sqrt{5}} sin\,x/\sqrt{5}\,.$$

Since $2x/\sqrt{5} = \,ln\,t$, the solution in the original variable t is

$$\begin{aligned} y(t) &= c_1 e^{-ln\,t} cos\left(ln\sqrt{t}\right) + c_2\, e^{-ln\,t} sin\left(ln\sqrt{t}\right) \\ &= t^{-1}\left[c_1 cos\left(ln\sqrt{t}\right) + c_2\, sin\left(ln\sqrt{t}\right)\right]. \end{aligned}$$

42. $p(t) = -4/t$ and $q(t) = -6/t^2$. Set $x = \sqrt{6}\int t^{-1}dt = \sqrt{6}\, ln\, t$.
Checking the feasibility of the transformation (**see Prob. 34 d, with* $q < 0$),

$$\frac{-q'(t) - 2p(t)q(t)}{2[-q(t)]^{3/2}} = \frac{-5}{\sqrt{6}}\,.$$

The ratio is constant, and therefore the equation can be transformed. In fact, we obtain

$$\frac{d^2y}{dx^2} + \frac{-5}{\sqrt{6}}\frac{dy}{dx} - y = 0\,.$$

Based on the methods in this section, the characteristic equation is $\sqrt{6}\, r^2 - 5$
$r - \sqrt{6} = 0$,
with roots $r = \sqrt{6}\,,\ -1/\sqrt{6}$. The general solution is

$$y(x) = c_1 e^{\sqrt{6}\,x} + c_2\, e^{-x/\sqrt{6}}.$$

Since $x = \sqrt{6}\, ln\, t$, the solution in the original variable t is

$$\begin{aligned} y(t) &= c_1 e^{6ln\,t} + c_2\, e^{-ln\,t} \\ &= c_1 t^6 + c_2\, t^{-1}. \end{aligned}$$

Section 3.5

2. The characteristic equation is $9r^2 + 6r + 1 = 0$, with the *double* root $r = -1/3$. Based on the discussion in this section, the general solution is $y(t) = c_1 e^{-t/3} + c_2 t\, e^{-t/3}$.

3. The characteristic equation is $4r^2 - 4r - 3 = 0$, with roots $r = -1/2\,, 3/2$. The general solution is $y(t) = c_1 e^{-t/2} + c_2 e^{3t/2}$.

4. The characteristic equation is $4r^2 + 12r + 9 = 0$, with the *double* root $r = -3/2$. Based on the discussion in this section, the general solution is $y(t) = (c_1 + c_2\, t)e^{-3t/2}$.

5. The characteristic equation is $r^2 - 2r + 10 = 0$, with complex roots $r = 1 \pm 3i$. The general solution is $y(t) = c_1 e^t cos\, 3t + c_2 e^t sin\, 3t$.

6. The characteristic equation is $r^2 - 6r + 9 = 0$, with the *double* root $r = 3$. The general solution is $y(t) = c_1 e^{3t} + c_2 t\, e^{3t}$.

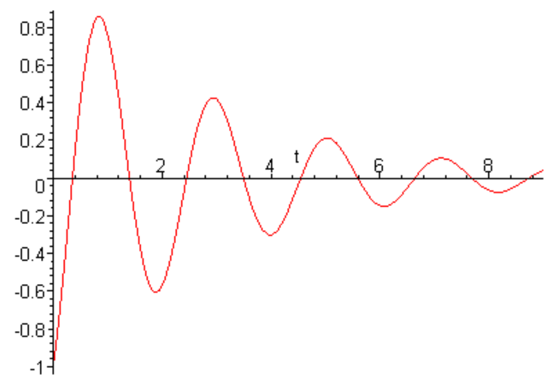
7. The characteristic equation is $4r^2 + 17r + 4 = 0$, with roots $r = -1/4\,, -4$. The general solution is $y(t) = c_1 e^{-t/4} + c_2 e^{-4t}$.

8. The characteristic equation is $16r^2 + 24r + 9 = 0$, with the *double* root $r = -3/4$. The general solution is $y(t) = c_1 e^{-3t/4} + c_2 t\, e^{-3t/4}$.

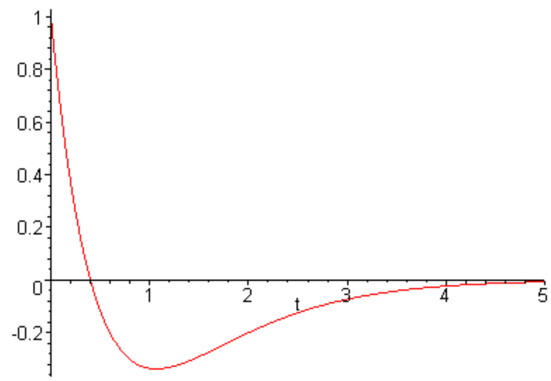
10. The characteristic equation is $2r^2 + 2r + 1 = 0$, with complex roots $r = -\frac{1}{2} \pm \frac{1}{2}\, i$. The general solution is $y(t) = c_1 e^{-t/2} cos\, t/2 + c_2 e^{-t/2} sin\, t/2$.

11. The characteristic equation is $9r^2 - 12r + 4 = 0$, with the *double* root $r = 2/3$. The general solution is $y(t) = c_1 e^{2t/3} + c_2 t\, e^{2t/3}$. Invoking the first initial condition, it follows that $c_1 = 2$. Now $y'(t) = (4/3 + c_2)e^{2t/3} + 2c_2 t\, e^{2t/3}/3$. Invoking the second initial condition, $4/3 + c_2 = -1$, or $c_2 = -7/3$. Hence $y(t) = 2e^{2t/3} - \frac{7}{3} t\, e^{2t/3}$. Since the *second* term dominates for large t, $y(t) \to -\infty$.

13. The characteristic equation is $9r^2 + 6r + 82 = 0$, with complex roots $r = -\frac{1}{3} \pm 3\, i$. The general solution is $y(t) = c_1 e^{-t/3} cos\, 3t + c_2 e^{-t/3} sin\, 3t$. Based on the first initial condition, $c_1 = -1$. Invoking the second initial condition, $1/3 + 3c_2 = 2$, or $c_2 = \frac{5}{9}$. Hence $y(t) = -e^{-t/3} cos\, 3t + \frac{5}{9} e^{-t/3} sin\, 3t$.



$15(a)$. The characteristic equation is $4r^2+12r+9=0$, with the *double* root $r=-\frac{3}{2}$. The general solution is $y(t)=c_1e^{-3t/2}+c_2t\,e^{-3t/2}$. Invoking the first initial condition, it follows that $c_1=1$. Now $y'(t)=(-3/2+c_2)e^{2t/3}-\frac{3}{2}c_2t\,e^{2t/3}$. The second initial condition requires that $-3/2+c_2=-4$, or $c_2=-5/2$. Hence the specific solution is $y(t)=e^{-3t/2}-\frac{5}{2}t\,e^{-3t/2}$.



(b). The solution crosses the *x-axis* at $t=0.4$.

(c). The solution has a minimum at the point $(16/15\,,\,-5e^{-8/5}/3)$.

(d). Given that $y'(0)=b$, we have $-3/2+c_2=b$, or $c_2=b+3/2$. Hence the solution is $y(t)=e^{-3t/2}+\left(b+\frac{3}{2}\right)t\,e^{-3t/2}$. Since the *second* term dominates, the *long-term* solution depends on the *sign* of the coefficient $b+\frac{3}{2}$. The critical value is $b=-\frac{3}{2}$.

16. The characteristic roots are $r_1=r_2=1/2$. Hence the general solution is given by $y(t)=c_1e^{t/2}+c_2t\,e^{t/2}$. Invoking the initial conditions, we require that $c_1=2$, and that $1+c_2=b$. The specific solution is $y(t)=2e^{t/2}+(b-1)t\,e^{t/2}$. Since the *second* term dominates, the *long-term* solution depends on the *sign* of the coefficient $b-1$. The critical value is $b=1$.

18(a). The characteristic roots are $r_1 = r_2 = -2/3$. Therefore the general solution is given by $y(t) = c_1 e^{-2t/3} + c_2 t\, e^{-2t/3}$. Invoking the initial conditions, we require that $c_1 = a$, and that $-2a/3 + c_2 = -1$. After solving for the coefficients, the specific solution is $y(t) = ae^{-2t/3} + \left(\frac{2a}{3} - 1\right)t\, e^{-2t/3}$.

(b). Since the *second* term dominates, the *long-term* solution depends on the *sign* of the coefficient $\frac{2a}{3} - 1$. The critical value is $a = 3/2$.

20(a). The characteristic equation is $r^2 + 2ar + a^2 = 0$, with *double* root $r = -a$. Hence one solution is $y_1(t) = c_1 e^{-at}$.

(b). Recall that the Wronskian satisfies the differential equation $W' + 2aW = 0$. The solution of this equation is $W(t) = c\, e^{-2at}$.

(c). By definition, $W = y_1 y_2' - y_1' y_2$. Hence $c_1 e^{-at} y_2' + ac_1 e^{-at} y_2 = c\, e^{-2at}$. That is, $y_2' + a\, y_2 = c_2 e^{-at}$. This equation is first order *linear*, with general solution $y_2(t) = c_2 te^{-at} + c_3 e^{-at}$. Setting $c_2 = 1$ and $c_3 = 0$, we obtain $y_2(t) = te^{-at}$.

22(a). Write $ar^2 + br + c = a\left(r^2 + \frac{b}{a}r + \frac{c}{a}\right)$. It follows that $\frac{b}{a} = -2r_1$ and $\frac{c}{a} = r_1^2$. Hence $ar^2 + br + c = ar^2 - 2ar_1 r + ar_1^2 = a(r^2 - 2r_1 r + r_1^2) = a(r - r_1)^2$. We find that $L[e^{rt}] = (ar^2 + br + c)e^{rt} = a(r - r_1)^2 e^{rt}$. Setting $r = r_1$, $L[e^{r_1 t}] = 0$.

(b). Differentiating Eq.(i) with respect to r,

$$\frac{\partial}{\partial r} L\left[e^{rt}\right] = ate^{rt}(r - r_1)^2 + 2ae^{rt}(r - r_1).$$

Now observe that

$$\begin{aligned}\frac{\partial}{\partial r} L\left[e^{rt}\right] &= \frac{\partial}{\partial r}\left[a\frac{\partial^2}{\partial t^2}\left(e^{rt}\right) + b\frac{\partial}{\partial t}\left(e^{rt}\right) + c\left(e^{rt}\right)\right] \\ &= \left[a\frac{\partial^2}{\partial t^2}\left(\frac{\partial}{\partial r}e^{rt}\right) + b\frac{\partial}{\partial t}\left(\frac{\partial}{\partial r}e^{rt}\right) + c\left(\frac{\partial}{\partial r}e^{rt}\right)\right] \\ &= a\frac{\partial^2}{\partial t^2}\left(te^{rt}\right) + b\frac{\partial}{\partial t}\left(te^{rt}\right) + c\left(te^{rt}\right).\end{aligned}$$

Hence $L[te^{rt}] = ate^{rt}(r - r_1)^2 + 2ae^{rt}(r - r_1)$. Setting $r = r_1$, $L[te^{r_1 t}] = 0$.

23. Set $y_2(t) = t^2 v(t)$. Substitution into the ODE results in

$$t^2\left(t^2 v'' + 4tv' + 2v\right) - 4t\left(t^2 v' + 2tv\right) + 6t^2 v = 0.$$

After collecting terms, we end up with $t^4 v'' = 0$. Hence $v(t) = c_1 + c_2 t$, and thus $y_2(t) = c_1 t^2 + c_2 t^3$. Setting $c_1 = 0$ and $c_2 = 1$, we obtain $y_2(t) = t^3$.

24. Set $y_2(t) = t\, v(t)$. Substitution into the ODE results in

$$t^2(tv'' + 2v') + 2t(tv' + v) - 2tv = 0\,.$$

After collecting terms, we end up with $t^3v'' + 4t^2v' = 0$. This equation is *linear* in the variable $w = v'$. It follows that $v'(t) = c\,t^{-4}$, and $v(t) = c_1t^{-3} + c_2$. Thus $y_2(t) = c_1t^{-2} + c_2t$. Setting $c_1 = 1$ and $c_2 = 0$, we obtain $y_2(t) = t^{-2}$.

26. Set $y_2(t) = t\,v(t)$. Substitution into the ODE results in $v'' - v' = 0$. This ODE is *linear* in the variable $w = v'$. It follows that $v'(t) = c_1e^t$, and $v(t) = c_1e^t + c_2$. Thus $y_2(t) = c_1te^t + c_2t$. Setting $c_1 = 1$ and $c_2 = 0$, we obtain $y_2(t) = te^t$.

28. Set $y_2(x) = e^xv(x)$. Substitution into the ODE results in

$$v'' + \frac{x-2}{x-1}v' = 0\,.$$

This ODE is *linear* in the variable $w = v'$. An integrating factor is

$$\begin{aligned}\mu &= exp\left(\int \frac{x-2}{x-1}dx\right)\\ &= \frac{e^x}{x-1}\,.\end{aligned}$$

Rewrite the equation as $\left[\frac{e^xv'}{x-1}\right]' = 0$, from which it follows that $v'(x) = c(x-1)e^{-x}$. Hence $v(x) = c_1xe^{-x} + c_2$ and $y_2(x) = c_1x + c_2e^x$. Setting $c_1 = 1$ and $c_2 = 0$, we obtain $y_2(x) = x$.

29. Set $y_2(x) = y_1(x)\,v(x)$, in which $y_1(x) = x^{1/4}exp(2\sqrt{x})$. It can be verified that y_1 is a solution of the ODE, that is, $x^2y_1'' - (x - 0.1875)y_1 = 0$. Substitution of the given form of y_2 results in the differential equation

$$2x^{9/4}v'' + \left(4x^{7/4} + x^{5/4}\right)v' = 0\,.$$

This ODE is *linear* in the variable $w = v'$. An integrating factor is

$$\begin{aligned}\mu &= exp\left(\int \left[2x^{-1/2} + \frac{1}{2x}\right]dx\right)\\ &= \sqrt{x}\;exp\left(4\sqrt{x}\right).\end{aligned}$$

Rewrite the equation as $\left[\sqrt{x}\;exp\left(4\sqrt{x}\right)\,v'\right]' = 0$, from which it follows that

$$v'(x) = c\,exp\left(-4\sqrt{x}\right)/\sqrt{x}\,.$$

Integrating, $v(x) = c_1exp\left(-4\sqrt{x}\right) + c_2$ and as a result,

$$y_2(x) = c_1x^{1/4}exp\left(-2\sqrt{x}\right) + c_2x^{1/4}exp\left(2\sqrt{x}\right).$$

Setting $c_1 = 1$ and $c_2 = 0$, we obtain $y_2(x) = x^{1/4}exp\left(-2\sqrt{x}\right)$.

32. Direct substitution verifies that $y_1(t) = exp(-\delta x^2/2)$ is a solution of the ODE. Now set $y_2(x) = y_1(x)\,v(x)$. Substitution of y_2 into the ODE results in

$$v'' - \delta x v' = 0\,.$$

This ODE is *linear* in the variable $w = v'$. An integrating factor is $\mu = exp(-\delta x^2/2)$. Rewrite the equation as $[\,exp(-\delta x^2/2)v']' = 0$, from which it follows that

$$v'(x) = c_1\,exp\big(\delta x^2/2\big)\,.$$

Integrating, we obtain

$$v(x) = c_1 \int_{x_0}^{x} exp\big(\delta u^2/2\big)du + v(x_0).$$

Hence

$$y_2(x) = c_1 exp\big(-\delta x^2/2\big)\int_{x_0}^{x} exp\big(\delta u^2/2\big)du + c_2 exp\big(-\delta x^2/2\big).$$

Setting $c_2 = 0$, we obtain a second independent solution.

34. After writing the ODE in standard form, we have $p(t) = 3/t$. Based on *Abel's identity*, $W(y_1, y_2) = c_1 exp\big(-\int \frac{3}{t}dt\big) = c_1 t^{-3}$. As shown in Prob. 33, two solutions of a second order linear equation satisfy

$$(y_2/y_1)' = W(y_1, y_2)/y_1^2\,.$$

In the given problem, $y_1(t) = t^{-1}$. Hence $(t\,y_2)' = c_1 t^{-1}$. Integrating both sides of the equation, $y_2(t) = c_1 t^{-1} ln\,t + c_2 t^{-1}$.

36. After writing the ODE in standard form, we have $p(x) = -x/(x-1)$. Based on *Abel's identity*, $W(y_1, y_2) = c\;exp\big(\int \frac{x}{x-1}dx\big) = c\;e^x(x-1)$. Two solutions of a second order linear equation satisfy

$$(y_2/y_1)' = W(y_1, y_2)/y_1^2\,.$$

In the given problem, $y_1(x) = e^x$. Hence $(e^{-x}\,y_2)' = c\,e^{-x}(x-1)$. Integrating both sides of the equation, $y_2(x) = c_1 x + c_2 e^x$. Setting $c_1 = 1$ and $c_2 = 0$, we obtain $y_2(x) = x$.

37. Write the ODE in standard form to find $p(x) = 1/x$. Based on *Abel's identity*, $W(y_1, y_2) = c\;exp\big(-\int \frac{1}{x}dx\big) = c\,x^{-1}$. Two solutions of a second order linear ODE satisfy $(y_2/y_1)' = W(y_1, y_2)/y_1^2$. In the given problem, $y_1(x) = x^{-1/2} sin\,x$. Hence

$$\left(\frac{\sqrt{x}}{sin\,x}\,y_2\right)' = c\,\frac{1}{sin^2 x}\,.$$

Integrating both sides of the equation, $y_2(x) = c_1 x^{-1/2} cos\, x + c_2 x^{-1/2} sin\, x$. Setting $c_1 = 1$ and $c_2 = 0$, we obtain $y_2(x) = x^{-1/2} cos\, x$.

$39(a)$. The characteristic equation is $ar^2 + c = 0$. If $a, c > 0$, then the roots are $r_{1,2} = \pm i\sqrt{c/a}$. The general solution is

$$y(t) = c_1 cos\sqrt{\frac{c}{a}}\, t + c_2 sin\sqrt{\frac{c}{a}}\, t\,,$$

which is bounded.

(b). The characteristic equation is $ar^2 + br = 0$. The roots are $r_{1,2} = 0\,,\, -b/a$, and hence the general solution is $y(t) = c_1 + c_2 exp(-bt/a)$. Clearly, $y(t) \to c_1$.

40. Note that $cos\, t\, sin\, t = \frac{1}{2} sin\, 2t$. So that $1 - k\, cos\, t\, sin\, t = 1 - \frac{k}{2} sin\, 2t$. If $0 < k < 2$, then $\frac{k}{2} sin\, 2t < |sin\, 2t|$ and $-\frac{k}{2} sin\, 2t > -|sin\, 2t|$. Hence

$$\begin{aligned} 1 - k\, cos\, t\, sin\, t &= 1 - \frac{k}{2} sin\, 2t \\ &> 1 - |sin\, 2t| \\ &\geq 0\,. \end{aligned}$$

41. $p(t) = -3/t$ and $q(t) = 4/t^2$. We have $x = 2\int t^{-1} dt = 2\, ln\, t$, and $t = e^{x/2}$. Furthermore,

$$\frac{q'(t) + 2p(t)q(t)}{2[q(t)]^{3/2}} = -2\,.$$

The ratio is constant, and therefore the equation can be transformed. In fact, we obtain

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0\,.$$

The general solution of this ODE is $y(x) = c_1 e^x + c_2 x e^x$. In terms of the original independent variable, $y(t) = c_1 t^2 + c_2 t^2\, ln\, t$.

Section 3.6

2. The characteristic equation for the homogeneous problem is $r^2 + 2r + 5 = 0$, with complex roots $r = -1 \pm 2i$. Hence $y_c(t) = c_1 e^{-t} cos\, 2t + c_2 e^{-t} sin\, 2t$. Since the function $g(t) = 3\, sin\, 2t$ is not proportional to the solutions of the homogeneous equation, set $Y = A\, cos\, 2t + B\, sin\, 2t$. Substitution into the given ODE, and comparing the coefficients, results in the system of equations $B - 4A = 3$ and $A + 4B = 0$. Hence $Y = -\frac{12}{17} cos\, 2t + \frac{3}{17} sin\, 2t$. The general solution is $y(t) = y_c(t) + Y$.

3. The characteristic equation for the homogeneous problem is $r^2 - 2r - 3 = 0$, with roots $r = -1, 3$. Hence $y_c(t) = c_1 e^{-t} + c_2 e^{3t}$. Note that the assignment $Y = Ate^{-t}$ is *not* sufficient to match the coefficients. Try $Y = Ate^{-t} + Bt^2 e^{-t}$. Substitution into the differential equation, and comparing the coefficients, results in the system of equations $-4A + 2B = 0$ and $-8B = -3$. Hence $Y = \frac{3}{16} te^{-t} + \frac{3}{8} t^2 e^{-t}$. The general
solution is $y(t) = y_c(t) + Y$.

5. The characteristic equation for the homogeneous problem is $r^2 + 9 = 0$, with complex roots $r = \pm 3i$. Hence $y_c(t) = c_1 cos\, 3t + c_2 sin\, 3t$. To simplify the analysis, set $g_1(t) = 6$ and $g_2(t) = t^2 e^{3t}$. By inspection, we have $Y_1 = 2/3$. Based on the form of g_2, set $Y_2 = Ae^{3t} + Bte^{3t} + Ct^2 e^{3t}$. Substitution into the differential equation, and comparing the coefficients, results in the system of equations $18A + 6B + 2C = 0$, $18B + 12C = 0$, and $18C = 1$. Hence

$$Y_2 = \frac{1}{162} e^{3t} - \frac{1}{27} te^{3t} + \frac{1}{18} t^2 e^{3t}.$$

The general solution is $y(t) = y_c(t) + Y_1 + Y_2$.

7. The characteristic equation for the homogeneous problem is $2r^2 + 3r + 1 = 0$, with roots $r = -1, -1/2$. Hence $y_c(t) = c_1 e^{-t} + c_2\, e^{-t/2}$. To simplify the analysis, set $g_1(t) = t^2$ and $g_2(t) = 3\, sin\, t$. Based on the form of g_1, set $Y_1 = A + Bt + Ct^2$. Substitution into the differential equation, and comparing the coefficients, results in the system of equations $A + 3B + 4C = 0$, $B + 6C = 0$, and $C = 1$. Hence we obtain $Y_1 = 14 - 6t + t^2$. On the other hand, set $Y_2 = D\, cos\, t + E\, sin\, t$. After substitution into the ODE, we find that $D = -9/10$ and $E = -3/10$. The general solution is $y(t) = y_c(t) + Y_1 + Y_2$.

9. The characteristic equation for the homogeneous problem is $r^2 + \omega_0^2 = 0$, with complex roots $r = \pm\, \omega_0 i$. Hence $y_c(t) = c_1 cos\, \omega_0 t + c_2 sin\, \omega_0 t$. Since $\omega \neq \omega_0$, set $Y = A\, cos\, \omega t + B\, sin\, \omega t$. Substitution into the ODE and comparing the coefficients results in the system of equations $(\omega_0^2 - \omega^2)A = 1$ and $(\omega_0^2 - \omega^2)B = 0$. Hence

$$Y = \frac{1}{\omega_0^2 - \omega^2}\, cos\, \omega t\, .$$

The general solution is $y(t) = y_c(t) + Y$.

10. From Prob. 9, $y_c(t) = c$. Since $cos\,\omega_0 t$ is a solution of the homogeneous problem, set $Y = At\,cos\,\omega_0 t + Bt\,sin\,\omega_0 t$. Substitution into the given ODE and comparing the coefficients results in $A = 0$ and $B = \frac{1}{2\omega_0}$. Hence the general solution is $y(t) = c_1 cos\,\omega_0 t + c_2 sin\,\omega_0 t + \frac{t}{2\omega_0} sin\,\omega_0 t$.

12. The characteristic equation for the homogeneous problem is $r^2 - r - 2 = 0$, with roots $r = -1, 2$. Hence $y_c(t) = c_1 e^{-t} + c_2\,e^{2t}$. Based on the form of the right hand side, that is, $cosh(2t) = (e^{2t} + e^{-2t})/2$, set $Y = At\,e^{2t} + Be^{-2t}$. Substitution into the given ODE and comparing the coefficients results in $A = 1/6$ and $B = 1/8$. Hence the general solution is $y(t) = c_1 e^{-t} + c_2\,e^{2t} + t\,e^{2t}/6 + e^{-2t}/8$.

14. The characteristic equation for the homogeneous problem is $r^2 + 4 = 0$, with roots $r = \pm 2i$. Hence $y_c(t) = c_1 cos\,2t + c_2 sin\,2t$. Set $Y_1 = A + Bt + Ct^2$. Comparing the coefficients of the respective terms, we find that $A = -1/8$, $B = 0$, $C = 1/4$. Now set $Y_2 = D\,e^t$, and obtain $D = 3/5$. Hence the general solution is

$$y(t) = c_1 cos\,2t + c_2 sin\,2t - 1/8 + t^2/4 + 3\,e^t/5\,.$$

Invoking the initial conditions, we require that $19/40 + c_1 = 0$ and $3/5 + 2c_2 = 2$. Hence $c_1 = -19/40$ and $c_2 = 7/10$.

15. The characteristic equation for the homogeneous problem is $r^2 - 2r + 1 = 0$, with a double root $r = 1$. Hence $y_c(t) = c_1 e^t + c_2 t\,e^t$. Consider $g_1(t) = t\,e^t$. Note that g_1 is a solution of the homogeneous problem. Set $Y_1 = At^2 e^t + Bt^3 e^t$ (the *first* term is not sufficient for a match). Upon substitution, we obtain $Y_1 = t^3 e^t/6$. By inspection, $Y_2 = 4$. Hence the general solution is $y(t) = c_1 e^t + c_2 t\,e^t + t^3 e^t/6 + 4$. Invoking the initial conditions, we require that $c_1 + 4 = 1$ and $c_1 + c_2 = 1$. Hence $c_1 = -3$ and $c_2 = 4$.

17. The characteristic equation for the homogeneous problem is $r^2 + 4 = 0$, with roots $r = \pm 2i$. Hence $y_c(t) = c_1 cos\,2t + c_2 sin\,2t$. Since the function $sin\,2t$ is a solution of the homogeneous problem, set $Y = At\,cos\,2t + Bt\,sin\,2t$. Upon substitution, we obtain $Y = -\frac{3}{4} t\,cos\,2t$. Hence the general solution is $y(t) = c_1 cos\,2t + c_2 sin\,2t - \frac{1}{4} t\,cos\,2t$. Invoking the initial conditions, we require that $c_1 = 2$ and $2c_2 - \frac{3}{4} = -1$. Hence $c_1 = 2$ and $c_2 = -1/8$.

18. The characteristic equation for the homogeneous problem is $r^2 + 2r + 5 = 0$, with complex roots $r = -1 \pm 2i$. Hence $y_c(t) = c_1 e^{-t} cos\,2t + c_2 e^{-t} sin\,2t$. Based on the form of $g(t)$, set $Y = At\,e^{-t} cos\,2t + Bt\,e^{-t} sin\,2t$. After comparing coefficients, we obtain $Y = t\,e^{-t} sin\,2t$. Hence the general solution is

$$y(t) = c_1 e^{-t} cos\,2t + c_2 e^{-t} sin\,2t + t\,e^{-t} sin\,2t\,.$$

Invoking the initial conditions, we require that $c_1 = 1$ and $-c_1 + 2c_2 = 0$. Hence $c_1 = 1$ and $c_2 = 1/2$.

20. The characteristic equation for the homogeneous problem is $r^2+1=0$, with complex roots $r=\pm\, i$. Hence $y_c(t)=c_1 cos\, t+c_2 sin\, t$. Let $g_1(t)=t\, sin\, t$ and $g_2(t)=t$. By inspection, it is easy to see that $Y_2(t)=1$. Based on the form of $g_1(t)$, set $Y_1(t)=At\, cos\, t+Bt\, sin\, t+Ct^2 cos\, t+Dt^2 sin\, t$. Substitution into the equation and comparing the coefficients results in $A=0$, $B=1/4$, $C=\,-1/4$, and $D=0$. Hence $Y(t)=1+\frac{1}{4}t\, sin\, t-\frac{1}{4}t^2 cos\, t$.

21. The characteristic equation for the homogeneous problem is $r^2-5r+6=0$, with roots $r=2\,,3$. Hence $y_c(t)=c_1e^{2t}+c_2e^{3t}$. Consider $g_1(t)=e^{2t}(3t+4)sin\, t$, and $g_2(t)=e^t cos\, 2t$. Based on the form of these functions on the right hand side of the ODE,
set $Y_2(t)=e^t(A_1 cos\, 2t+A_2 sin\, 2t)$, $Y_1(t)=(B_1+B_2t)e^{2t}sin\, t+(C_1+C_2t)e^{2t}cos\, t$.
Substitution into the equation and comparing the coefficients results in

$$Y(t)=\,-\frac{1}{20}\left(e^t cos\, 2t+3e^t sin\, 2t\right)+\frac{3}{2}te^{2t}(cos\, t-sin\, t)\,+e^{2t}\left(\frac{1}{2}cos\, t-5sin\, t\right).$$

23. The characteristic roots are $r=2\,,2$. Hence $y_c(t)=c_1e^{2t}+c_2te^{2t}$. Consider the functions $g_1(t)=2t^2$, $g_2(t)=4te^{2t}$, and $g_3(t)=t\, sin\, 2t$. The corresponding forms of the respective parts of the particular solution are $Y_1(t)=A_0+A_1t+A_2t^2$, $Y_2(t)=$ $=e^{2t}(B_2t^2+B_3t^3)$, and $Y_3(t)=t(C_1 cos\, 2t+C_2 sin\, 2t)+(D_1 cos2t+D_2 sin2t)$.
Substitution into the equation and comparing the coefficients results in

$$Y(t)=\frac{1}{4}\left(3+4t+2t^2\right)+\frac{2}{3}t^3e^{2t}+\frac{1}{8}t\, cos\, 2t+\frac{1}{16}(cos\, 2t-sin\, 2t)\,.$$

24. The homogeneous solution is $y_c(t)=c_1 cos\, 2t+c_2 sin\, 2t$. Since $cos\, 2t$ and $sin\, 2t$ are both solutions of the homogeneous equation, set

$$Y(t)=t\left(A_0+A_1t+A_2t^2\right)cos\, 2t+t\left(B_0+B_1t+B_2t^2\right)sin\, 2t\,.$$

Substitution into the equation and comparing the coefficients results in

$$Y(t)=\left(\frac{13}{32}t-\frac{1}{12}t^3\right)cos\, 2t+\frac{1}{16}\left(28t+13t^2\right)sin\, 2t\,.$$

25. The homogeneous solution is $y_c(t)=c_1e^{-t}+c_2te^{-2t}$. None of the functions on the right hand side are solutions of the homogenous equation. In order to include all possible combinations of the derivatives, consider $Y(t)=e^t(A_0+A_1t+A_2t^2)cos\, 2t+$ $+\,e^t(B_0+B_1t+B_2t^2)sin\, 2t+e^{-t}(C_1 cos\, t+C_2 sin\, t)+De^t$. Substitution into the differential equation and comparing the coefficients results in

$$\begin{aligned}Y(t)\, &=e^t\left(A_0+A_1t+A_2t^2\right)cos\, 2t+\;+e^t\left(B_0+B_1t+B_2t^2\right)sin\, 2t+\\ &+e^{-t}\left(-\tfrac{2}{3}\,cos\, t+\tfrac{2}{3}\,sin\, t\right)+2e^t/3\,,\end{aligned}$$

in which $A_0 = -4105/35152$, $A_1 = 73/676$, $A_2 = -5/52$, $B_0 = -1233/35152$, $B_1 = 10/169$, $B_2 = 1/52$.

26. The homogeneous solution is $y_c(t) = c_1 e^{-t} cos\, 2t + c_2 e^{-t} sin\, 2t$. None of the terms on the right hand side are solutions of the homogenous equation. In order to include the appropriate combinations of derivatives, consider $Y(t) = e^{-t}(A_1 t + A_2 t^2) cos\, 2t +$ $+ e^{-t}(B_1 t + B_2 t^2) sin\, 2t + e^{-2t}(C_0 + C_1 t) cos\, 2t + e^{-2t}(D_0 + D_1 t) sin\, 2t$. Substitution into the differential equation and comparing the coefficients results in

$$\begin{aligned} Y(t) &= \frac{3}{16} t e^{-t} cos\, 2t + \frac{3}{8} t^2 e^{-t} sin\, 2t - \frac{1}{25} e^{-2t}(7 + 10t) cos\, 2t + \\ &+ \frac{1}{25} e^{-2t}(1 + 5t) sin\, 2t\,. \end{aligned}$$

27. The homogeneous solution is $y_c(t) = c_1 cos\, \lambda t + c_2 sin\, \lambda t$. Since the differential operator does not contain a *first derivative* (and $\lambda \neq m\pi$), we can set

$$Y(t) = \sum_{m=1}^{N} C_m sin\, m\pi t\,.$$

Substitution into the ODE yields

$$-\sum_{m=1}^{N} m^2 \pi^2 C_m sin\, m\pi t + \lambda^2 \sum_{m=1}^{N} C_m sin\, m\pi t = \sum_{m=1}^{N} a_m sin\, m\pi t\,.$$

Equating coefficients of the individual terms, we obtain

$$C_m = \frac{a_m}{\lambda^2 - m^2 \pi^2}\,,\ m = 1, 2 \cdots N.$$

29. The homogeneous solution is $y_c(t) = c_1 e^{-t} cos\, 2t + c_2 e^{-t} sin\, 2t$. The input function is *independent* of the homogeneous solutions, on any interval. Since the right hand side is
piecewise constant, it follows by inspection that

$$Y(t) = \begin{cases} 1/5\,, & 0 \le t \le \pi/2 \\ 0\ \ , & t > \pi/2 \end{cases}\,.$$

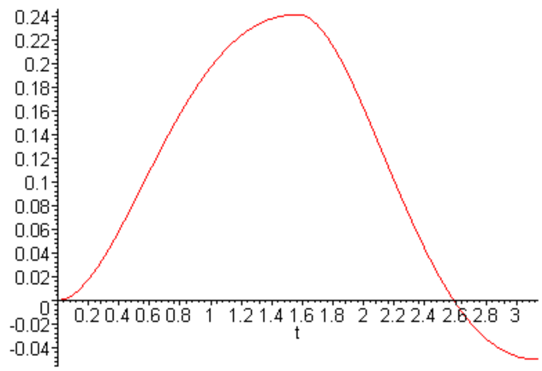
For $0 \le t \le \pi/2$, the general solution is $y(t) = c_1 e^{-t} cos\, 2t + c_2 e^{-t} sin\, 2t + 1/5$. Invoking the initial conditions $y(0) = y'(0) = 0$, we require that $c_1 = -1/5$, and that $c_2 = -1/10$. Hence

$$y(t) = \frac{1}{5} - \frac{1}{10}\left(2e^{-t} cos\, 2t + e^{-t} sin\, 2t\right)$$

on the interval $0 \le t \le \pi/2$. We now have the values $y(\pi/2) = \left(1 + e^{-\pi/2}\right)/5$, and $y'(\pi/2) = 0$. For $t > \pi/2$, the general solution is $y(t) = d_1 e^{-t} cos\, 2t + d_2 e^{-t} sin\, 2t$. It follows that $y(\pi/2) = -e^{-\pi/2} d_1$ and $y'(\pi/2) = e^{-\pi/2} d_1 - 2e^{-\pi/2} d_2$. Since the solution is continuously differentiable, we require that

$$-e^{-\pi/2}d_1 = \left(1 + e^{-\pi/2}\right)/5$$
$$e^{-\pi/2}d_1 - 2e^{-\pi/2}d_2 = 0\,.$$

Solving for the coefficients, $d_1 = 2d_2 = -\left(e^{\pi/2} + 1\right)/5$.



31. Since $a\,,b,c > 0$, the roots of the characteristic equation has *negative* real parts. That is, $r = \alpha \pm \beta\,i$, where $\alpha < 0$. Hence the homogeneous solution is

$$y_c(t) = c_1 e^{\alpha t} cos\,\beta t + c_2 e^{\alpha t} sin\,\beta t\,.$$

If $g(t) = d$, then the general solution is

$$y(t) = d/c + c_1 e^{\alpha t} cos\,\beta t + c_2 e^{\alpha t} sin\,\beta t\,.$$

Since $\alpha < 0$, $y(t) \to d/c$ as $t \to \infty$. If $c = 0$, then that characteristic roots are $r = 0$ and $r = -b/a$. The ODE becomes $ay'' + by' = d$. Integrating both sides, we find that $ay' + by = d\,t + c_1$. The general solution can be expressed as

$$y(t) = d\,t/b + c_1 + c_2 e^{-bt/a}.$$

In this case, the solution grows without bound. If $b = 0$, *also*, then the differential equation
can be written as $y'' = d/a$, which has general solution $y(t) = d\,t^2/2a + c_1 + c_2$. Hence the assertion is true only if the coefficients are *positive*.

32(a). Since D is a linear operator,

$$\begin{aligned} D^2y + bDy + cy &= D^2y - (r_1 + r_2)Dy + r_1 r_2 y \\ &= D^2y - r_2 Dy - r_1 Dy + r_1 r_2 y \\ &= D(Dy - r_2 y) - r_1(Dy - r_2 y) \\ &= (D - r_1)(D - r_2)y\,. \end{aligned}$$

(b). Let $u = (D - r_2)y$. Then the ODE (i) can be written as $(D - r_1)u = g(t)$, that is,

$u' - r_1 u = g(t)$. The latter is a linear *first order* equation in u. Its general solution is

$$u(t) = e^{r_1 t} \int_{t_0}^{t} e^{-r_1 \tau} g(\tau) d\tau + c_1 e^{r_1 t} .$$

From above, we have $y' - r_2 y = u(t)$. This equation is also a first order ODE. Hence the general solution of the original second order equation is

$$y(t) = e^{r_2 t} \int_{t_0}^{t} e^{-r_2 \tau} u(\tau) d\tau + c_2 e^{r_2 t} .$$

Note that the solution $y(t)$ contains *two* arbitrary constants.

34. Note that $(2D^2 + 3D + 1)y = (2D + 1)(D + 1)y$. Let $u = (D + 1)y$, and solve the ODE $2u' + u = t^2 + 3sin\, t$. This equation is a linear first order ODE, with solution

$$\begin{aligned} u(t) &= e^{-t/2} \int_{t_0}^{t} e^{\tau/2} \left[\tau^2/2 + \frac{3}{2} sin\, \tau\right] d\tau + c\, e^{-t/2} \\ &= t^2 - 4t + 8 - \frac{6}{5} cos\, t + \frac{3}{5} sin\, t + c\, e^{-t/2}. \end{aligned}$$

Now consider the ODE $y' + y = u(t)$. The general solution of this first order ODE is

$$y(t) = e^{-t} \int_{t_0}^{t} e^{\tau} u(\tau) d\tau + c_2 e^{-t} ,$$

in which $u(t)$ is given above. Substituting for $u(t)$ and performing the integration,

$$y(t) = t^2 - 6t + 14 - \frac{9}{10} cos\, t - \frac{3}{10} sin\, t + c_1 e^{-t/2} + c_2 e^{-t}.$$

35. We have $(D^2 + 2D + 1)y = (D + 1)(D + 1)y$. Let $u = (D + 1)y$, and consider the ODE $u' + u = 2e^{-t}$. The general solution is $u(t) = 2t\, e^{-t} + c\, e^{-t}$. We therefore have the first order equation $u' + u = 2t\, e^{-t} + c_1 e^{-t}$. The general solution of the latter differential equation is

$$\begin{aligned} y(t) &= e^{-t} \int_{t_0}^{t} [2\tau + c_1] d\tau + c_2 e^{-t} \\ &= e^{-t} \left(t^2 + c_1 t + c_2\right). \end{aligned}$$

36. We have $(D^2 + 2D)y = D(D + 2)y$. Let $u = (D + 2)y$, and consider the equation $u' = 3 + 4sin\, 2t$. Direct integration results in $u(t) = 3t - 2cos\, 2t + c$. The problem is reduced to solving the ODE $y' + 2y = 3t - 2cos\, 2t + c$. The general solution of this first order differential equation is

$$
\begin{aligned}
y(t) &= e^{-2t}\int_{t_0}^{t} e^{2\tau}[3\tau - 2cos\,2\tau + c\,]d\tau + c_2e^{-2t} \\
&= \frac{3}{2}t - \frac{1}{2}(cos\,2t + sin\,2t) + c_1 + c_2e^{-2t}.
\end{aligned}
$$

Section 3.7

1. The solution of the homogeneous equation is $y_c(t) = c_1e^{2t} + c_2e^{3t}$. The functions $y_1(t) = e^{2t}$ and $y_2(t) = e^{3t}$ form a fundamental set of solutions. The Wronskian of these functions is $W(y_1, y_2) = e^{5t}$. Using the method of *variation of parameters*, the particular solution is given by $Y(t) = u_1(t)\,y_1(t) + u_2(t)\,y_2(t)$, in which

$$\begin{aligned} u_1(t) &= -\int \frac{e^{3t}(2e^t)}{W(t)}dt \\ &= 2e^{-t} \end{aligned}$$

$$\begin{aligned} u_2(t) &= \int \frac{e^{2t}(2e^t)}{W(t)}dt \\ &= -\,e^{-2t} \end{aligned}$$

Hence the particular solution is $Y(t) = 2e^t - e^t = e^t$.

3. The solution of the homogeneous equation is $y_c(t) = c_1e^{-t} + c_2te^{-t}$. The functions $y_1(t) = e^{-t}$ and $y_2(t) = te^{-t}$ form a fundamental set of solutions. The Wronskian of these functions is $W(y_1, y_2) = e^{-2t}$. Using the method of *variation of parameters*, the particular solution is given by $Y(t) = u_1(t)\,y_1(t) + u_2(t)\,y_2(t)$, in which

$$\begin{aligned} u_1(t) &= -\int \frac{te^{-t}(3e^{-t})}{W(t)}dt \\ &= -\,3t^2/2 \end{aligned}$$

$$\begin{aligned} u_2(t) &= \int \frac{e^{-t}(3e^{-t})}{W(t)}dt \\ &= 3t \end{aligned}$$

Hence the particular solution is $Y(t) = -\,3t^2e^{-t}/2 + 3t^2e^{-t} = 3t^2e^{-t}/2$.

4. The functions $y_1(t) = e^{t/2}$ and $y_2(t) = te^{t/2}$ form a fundamental set of solutions. The Wronskian of these functions is $W(y_1, y_2) = e^t$. First write the equation in standard form, so that $g(t) = 4e^{t/2}$. Using the method of *variation of parameters*, the particular solution is given by $Y(t) = u_1(t)\,y_1(t) + u_2(t)\,y_2(t)$, in which

$$\begin{aligned} u_1(t) &= -\int \frac{te^{t/2}\left(4e^{t/2}\right)}{W(t)}dt \\ &= -\,2t^2 \end{aligned}$$

$$u_2(t) = \int \frac{e^{t/2}(4e^{t/2})}{W(t)}dt \\ = 4t$$

Hence the particular solution is $Y(t) = -2t^2e^{t/2} + 4t^2e^{t/2} = 2t^2e^{t/2}$.

6. The solution of the homogeneous equation is $y_c(t) = c_1 cos\,3t + c_2 sin\,3t$. The two functions $y_1(t) = cos\,3t$ and $y_2(t) = sin\,3t$ form a fundamental set of solutions, with $W(y_1, y_2) = 3$. The particular solution is given by $Y(t) = u_1(t)\,y_1(t) + u_2(t)\,y_2(t)$, in which

$$u_1(t) = -\int \frac{sin\,3t(9\,sec^2 3t)}{W(t)}dt \\ = -csc\,3t$$

$$u_2(t) = \int \frac{cos\,3t(9\,sec^2 3t)}{W(t)}dt \\ = ln|sec\,3t + tan\,3t|$$

Hence the particular solution is $Y(t) = -1 + (sin\,3t)ln|sec\,3t + tan\,3t|$. The general solution is given by $y(t) = c_1 cos\,3t + c_2 sin\,3t + (sin\,3t)ln|sec\,3t + tan\,3t| - 1$.

7. The functions $y_1(t) = e^{-2t}$ and $y_2(t) = te^{-2t}$ form a fundamental set of solutions. The Wronskian of these functions is $W(y_1, y_2) = e^{-4t}$. The particular solution is given by $Y(t) = u_1(t)\,y_1(t) + u_2(t)\,y_2(t)$, in which

$$u_1(t) = -\int \frac{te^{-2t}(t^{-2}e^{-2t})}{W(t)}dt \\ = -ln\,t$$

$$u_2(t) = \int \frac{e^{-2t}(t^{-2}e^{-2t})}{W(t)}dt \\ = -1/t$$

Hence the particular solution is $Y(t) = -e^{-2t}ln\,t - e^{-2t}$. Since the *second term* is a solution of the homogeneous equation, the general solution is given by $y(t) = c_1e^{-2t} + c_2te^{-2t} - e^{-2t}ln\,t$.

8. The solution of the homogeneous equation is $y_c(t) = c_1 cos\,2t + c_2 sin\,2t$. The two functions $y_1(t) = cos\,2t$ and $y_2(t) = sin\,2t$ form a fundamental set of solutions, with $W(y_1, y_2) = 2$. The particular solution is given by $Y(t) = u_1(t)\,y_1(t) + u_2(t)\,y_2(t)$, in which

$$u_1(t) = -\int \frac{sin\,2t(3\,csc\,2t)}{W(t)}dt$$
$$= -3t/2$$

$$u_2(t) = \int \frac{cos\,2t(3\,csc\,2t)}{W(t)}dt$$
$$= \frac{3}{4}ln|sin\,2t|$$

Hence the particular solution is $Y(t) = -\frac{3}{2}tcos\,2t + \frac{3}{4}(sin\,3t)ln|sin\,2t|$. The general solution is given by $y(t) = c_1 cos\,2t + c_2 sin\,2t - \frac{3}{2}tcos\,2t + \frac{3}{4}(sin\,3t)ln|sin\,2t|$.

9. The functions $y_1(t) = cos\,(t/2)$ and $y_2(t) = sin(t/2)$ form a fundamental set of solutions. The Wronskian of these functions is $W(y_1, y_2) = 1/2$. First write the ODE in standard form, so that $g(t) = sec(t/2)/2$. The particular solution is given by $Y(t) = u_1(t)\,y_1(t) + u_2(t)\,y_2(t)$, in which

$$u_1(t) = -\int \frac{cos\,(t/2)[sec(t/2)]}{2W(t)}dt$$
$$= 2\,ln[cos\,(t/2)]$$

$$u_2(t) = \int \frac{sin(t/2)[sec(t/2)]}{2W(t)}dt$$
$$= t$$

The particular solution is $Y(t) = 2cos(t/2)ln[cos\,(t/2)] + t\,sin(t/2)$. The general solution is given by

$$y(t) = c_1 cos\,(t/2) + c_2 sin(t/2) + 2\,cos(t/2)\,ln[cos\,(t/2)] + t\,sin(t/2).$$

10. The solution of the homogeneous equation is $y_c(t) = c_1 e^t + c_2 te^t$. The functions $y_1(t) = e^t$ and $y_2(t) = te^t$ form a fundamental set of solutions, with $W(y_1, y_2) = e^{2t}$. The particular solution is given by $Y(t) = u_1(t)\,y_1(t) + u_2(t)\,y_2(t)$, in which

$$u_1(t) = -\int \frac{te^t(e^t)}{W(t)(1+t^2)}dt$$
$$= -\frac{1}{2}ln\left(1+t^2\right)$$

$$u_2(t) = \int \frac{e^t(e^t)}{W(t)(1+t^2)}dt$$
$$= arctan\,t$$

The particular solution is $Y(t) = -\frac{1}{2}e^t\,ln(1+t^2) + te^t\,arctan(t)$. Hence the general solution is given by $y(t) = c_1 e^t + c_2 te^t - \frac{1}{2} e^t \, ln(1+t^2) + te^t \, arctan(t)$.

12. The functions $y_1(t) = cos\, 2t$ and $y_2(t) = sin\, 2t$ form a fundamental set of solutions, with $W(y_1, y_2) = 2$. The particular solution is given by $Y(t) = u_1(t)$ $y_1(t) + u_2(t)\, y_2(t)$,
in which

$$u_1(t) = -\frac{1}{2}\int^t g(s)\, sin\, 2s\, ds$$

$$u_2(t) = \frac{1}{2}\int^t g(s)\, cos\, 2s\, ds$$

Hence the particular solution is

$$Y(t) = -\frac{1}{2} cos\, 2t \int^t g(s)\, sin\, 2s\, ds + \frac{1}{2} sin\, 2t \int^t g(s)\, cos\, 2s\, ds\,.$$

Note that $sin\, 2t\, cos\, 2s - cos\, 2t\, sin\, 2s = sin(2t - 2s)$. It follows that

$$Y(t) = \frac{1}{2}\int^t g(s) sin(2t - 2s) ds\,.$$

The general solution of the differential equation is given by

$$y(t) = c_1 cos\, 2t + c_2 sin\, 2t + \frac{1}{2}\int^t g(s) sin(2t - 2s) ds\,.$$

13. Note first that $p(t) = 0\,, q(t) = -2/t^2$ and $g(t) = (3t^2 - 1)/t^2$. The functions $y_1(t)$ and $y_2(t)$ are solutions of the homogeneous equation, verified by substitution. The Wronskian of these two functions is $W(y_1, y_2) = -3$. Using the method of *variation of parameters*, the particular solution is $Y(t) = u_1(t)\, y_1(t) + u_2(t)\, y_2(t)$, in which

$$\begin{aligned} u_1(t) &= -\int \frac{t^{-1}(3t^2 - 1)}{t^2\, W(t)} dt \\ &= t^{-2}/6 + ln\, t \end{aligned}$$

$$\begin{aligned} u_2(t) &= \int \frac{t^2(3t^2 - 1)}{t^2\, W(t)} dt \\ &= -t^3/3 + t/3 \end{aligned}$$

Therefore $Y(t) = 1/6 + t^2 ln\, t - t^2/3 + 1/3$. Hence the general solution is

$$y(t) = c_1 t^2 + c_2 t^{-1} + t^2 ln\, t + 1/2\,.$$

15. Observe that $g(t) = t\,e^{2t}$. The functions $y_1(t)$ and $y_2(t)$ are a fundamental set of solutions. The Wronskian of these two functions is $W(y_1,y_2) = t\,e^t$. Using the method of *variation of parameters*, the particular solution is $Y(t) = u_1(t)\,y_1(t) + u_2(t)\,y_2(t)$, in which

$$\begin{aligned} u_1(t) &= -\int \frac{e^t(t\,e^{2t})}{W(t)}dt \\ &= -\,e^{2t}/2 \end{aligned}$$

$$\begin{aligned} u_2(t) &= \int \frac{(1+t)(t\,e^{2t})}{W(t)}dt \\ &= t\,e^t \end{aligned}$$

Therefore $Y(t) = -\,(1+t)e^{2t}/2 + t\,e^{2t} = -\,e^{2t}/2 + t\,e^{2t}/2\,.$

16. Observe that $g(t) = 2(1-t)\,e^{-t}$. Direct substitution of $y_1(t) = e^t$ and $y_2(t) = t$ verifies that they are solutions of the homogeneous equation. The Wronskian of the two solutions is $W(y_1,y_2) = (1-t)\,e^t$. Using the method of *variation of parameters*, the particular solution is $Y(t) = u_1(t)\,y_1(t) + u_2(t)\,y_2(t)$, in which

$$\begin{aligned} u_1(t) &= -\int \frac{2t(1-t)e^{-t}}{W(t)}dt \\ &= te^{-2t} + e^{-2t}/2 \end{aligned}$$

$$\begin{aligned} u_2(t) &= \int \frac{2(1-t)}{W(t)}dt \\ &= -\,2\,e^{-t} \end{aligned}$$

Therefore $Y(t) = te^{-t} + e^{-t}/2 - 2t\,e^{-t} = -\,te^{-t} + \,e^{-t}/2\,.$

17. Note that $g(x) = ln\,x$. The functions $y_1(x) = x^2$ and $y_2(x) = x^2 ln\,x$ are solutions of the homogeneous equation, as verified by substitution. The Wronskian of the solutions is $W(y_1,y_2) = x^3$. Using the method of *variation of parameters*, the particular solution is

$$Y(x) = u_1(x)\,y_1(x) + u_2(x)\,y_2(x),$$

in which

$$\begin{aligned} u_1(x) &= -\int \frac{x^2 ln\,x(ln\,x)}{W(x)}dx \\ &= -\,(ln\,x)^3/3 \end{aligned}$$

$$u_2(x) = \int \frac{x^2(ln\,x)}{W(x)}dx$$
$$= (ln\,x)^2/2$$

Therefore $Y(x) = -x^2(ln\,x)^3/3 + x^2(ln\,x)^3/2 = x^2(ln\,x)^3/6$.

19. First write the equation in *standard form*. Note that the forcing function becomes $g(x)/(1-x)$. The functions $y_1(x) = e^x$ and $y_2(x) = x$ are a fundamental set of solutions,
as verified by substitution. The Wronskian of the solutions is $W(y_1,y_2) = (1-x)e^x$. Using the method of *variation of parameters*, the particular solution is

$$Y(x) = u_1(x)\,y_1(x) + u_2(x)\,y_2(x),$$

in which

$$u_1(x) = -\int^x \frac{\tau(g(\tau))}{(1-\tau)W(\tau)}d\tau$$

$$u_2(x) = \int^x \frac{e^\tau(g(\tau))}{(1-\tau)W(\tau)}d\tau$$

Therefore

$$Y(x) = -e^x\int^x \frac{\tau(g(\tau))}{(1-\tau)W(\tau)}d\tau + x\int^x \frac{e^\tau(g(\tau))}{(1-\tau)W(\tau)}d\tau$$
$$= \int^x \frac{(xe^\tau - e^x\tau)g(\tau)}{(1-\tau)^2e^\tau}d\tau\,.$$

20. First write the equation in *standard form*. The forcing function becomes $g(x)/x^2$. The functions $y_1(x) = x^{-1/2}sin\,x$ and $y_2(x) = x^{-1/2}cos\,x$ are a fundamental set of solutions. The Wronskian of the solutions is $W(y_1,y_2) = -1/x$. Using the method of *variation of parameters*, the particular solution is

$$Y(x) = u_1(x)\,y_1(x) + u_2(x)\,y_2(x),$$

in which

$$u_1(x) = \int^x \frac{cos\,\tau\,(g(\tau))}{\tau\sqrt{\tau}}d\tau$$

$$u_2(x) = -\int^x \frac{sin\,\tau\,(g(\tau))}{\tau\sqrt{\tau}}d\tau$$

Therefore

$$
\begin{aligned}
Y(x) &= \frac{sin\,x}{\sqrt{x}}\int^{x}\frac{cos\,\tau\,(g(\tau))}{\tau\sqrt{\tau}}dt - \frac{cos\,x}{\sqrt{x}}\int^{x}\frac{sin\,\tau\,(g(\tau))}{\tau\sqrt{\tau}}d\tau \\
&= \frac{1}{\sqrt{x}}\int^{x}\frac{sin(x-\tau)\,g(\tau)}{\tau\sqrt{\tau}}d\tau\,.
\end{aligned}
$$

21. Let $y_1(t)$ and $y_2(t)$ be a fundamental set of solutions, and $W(t) = W(y_1, y_2)$ be the corresponding Wronskian. Any solution, $u(t)$, of the homogeneous equation is a linear combination $u(t) = \alpha_1 y_1(t) + \alpha_2 y_2(t)$. Invoking the initial conditions, we require that

$$
\begin{aligned}
y_0 &= \alpha_1\, y_1(t_0) + \alpha_2\, y_2(t_0) \\
y_0' &= \alpha_1\, y_1'(t_0) + \alpha_2\, y_2'(t_0)
\end{aligned}
$$

Note that this system of equations has a unique solution, since $W(t_0) \neq 0$. Now consider the *nonhomogeneous* problem, $L[v] = g(t)$,with *homogeneous* initial conditions. Using the method of variation of parameters, the particular solution is given by

$$
Y(t) = \, - y_1(t)\int_{t_0}^{t}\frac{y_2(s)\,g(s)}{W(s)}ds + y_2(t)\int_{t_0}^{t}\frac{y_1(s)\,g(s)}{W(s)}ds\,.
$$

The general solution of the IVP (iii) is

$$
\begin{aligned}
v(t) &= \beta_1 y_1(t) + \beta_2 y_2(t) + Y(t) \\
&= \beta_1 y_1(t) + \beta_2 y_2(t) + y_1(t)u_1(t) + y_2(t)u_2(t)
\end{aligned}
$$

in which u_1 and u_2 are defined above. Invoking the initial conditions, we require that

$$
\begin{aligned}
0 &= \beta_1 y_1(t_0) + \beta_2 y_2(t_0) + Y(t_0) \\
0 &= \beta_1 y_1'(t_0) + \beta_2 y_2'(t_0) + Y'(t_0)
\end{aligned}
$$

Based on the definition of u_1 and u_2 , $Y(t_0) = 0$. Furthermore, since $y_1 u_1' + y_2 u_2' = 0$, it follows that $Y'(t_0) = 0$. Hence the only solution of the above system of equations is the *trivial solution*. Therefore $v(t) = Y(t)$. Now consider the function $y = u + v$. Then $L[y] = L[u+v] = L[u] + L[v] = g(t)$. That is, $y(t)$ is a solution of the nonhomogeneous
problem. Further, $y(t_0) = u(t_0) + v(t_0) = y_0$, and similarly, $y'(t_0) = y_0'$. By the uniqueness theorems, $y(t)$ is the unique solution of the initial value problem.

23. A fundamental set of solutions is $y_1(t) = cos\,t$ and $y_2(t) = sin\,t$. The Wronskian $W(t) = y_1 y_2' - y_1' y_2 = 1$. By the result in Prob. 22,

$$
\begin{aligned}
Y(t) &= \int_{t_0}^{t}\frac{cos(s)\,sin(t) - cos(t)\,sin(s)}{W(s)}g(s)ds \\
&= \int_{t_0}^{t}[cos(s)\,sin(t) - cos(t)\,sin(s)]g(s)ds\,.
\end{aligned}
$$

Finally, we have $cos(s)\,sin(t) - cos(t)\,sin(s) = sin(t-s)$.

24. A fundamental set of solutions is $y_1(t) = e^{at}$ and $y_2(t) = e^{bt}$. The Wronskian $W(t) = y_1 y_2' - y_1' y_2 = (b-a)exp[(a+b)t]$. By the result in Prob. 22,

$$\begin{aligned} Y(t) &= \int_{t_0}^{t} \frac{e^{as}e^{bt} - e^{at}e^{bs}}{W(s)} g(s)ds \\ &= \frac{1}{b-a}\int_{t_0}^{t} \frac{e^{as}e^{bt} - e^{at}e^{bs}}{exp[(a+b)s]} g(s)ds \,. \end{aligned}$$

Hence the particular solution is

$$Y(t) = \frac{1}{b-a}\int_{t_0}^{t} \left[e^{b(t-s)} - e^{a(t-s)}\right] g(s)ds \,.$$

26. A fundamental set of solutions is $y_1(t) = e^{at}$ and $y_2(t) = te^{at}$. The Wronskian $W(t) = y_1 y_2' - y_1' y_2 = e^{2at}$. By the result in Prob. 22,

$$\begin{aligned} Y(t) &= \int_{t_0}^{t} \frac{e^{as}e^{bt} - e^{at}e^{bs}}{W(s)} g(s)ds \\ &= \frac{1}{b-a}\int_{t_0}^{t} \frac{e^{as}e^{bt} - e^{at}e^{bs}}{exp[(a+b)s]} g(s)ds \,. \end{aligned}$$

Hence the particular solution is

$$Y(t) = \frac{1}{b-a}\int_{t_0}^{t} \left[e^{b(t-s)} - e^{a(t-s)}\right] g(s)ds \,.$$

26. A fundamental set of solutions is $y_1(t) = e^{at}$ and $y_2(t) = te^{at}$. The Wronskian $W(t) = y_1 y_2' - y_1' y_2 = e^{2at}$. By the result in Prob. 22,

$$\begin{aligned} Y(t) &= \int_{t_0}^{t} \frac{te^{as+at} - s\,e^{at+as}}{W(s)} g(s)ds \\ &= \int_{t_0}^{t} \frac{(t-s)e^{as+at}}{e^{2as}} g(s)ds \,. \end{aligned}$$

Hence the particular solution is

$$Y(t) = \int_{t_0}^{t} (t-s)e^{a(t-s)} g(s)ds \,.$$

27. Depending on the values of a, b and c, the operator $aD^2 + bD + c$ can have *three* types of fundamental solutions.

(i) The characteristic roots $r_{1,2} = \alpha\,,\beta\,;\ \alpha \neq \beta\,.$ $y_1(t) = e^{\alpha t}$ and $y_2(t) = e^{\beta t}$.

$$K(t) = \frac{1}{\beta - \alpha}\left[e^{\beta t} - e^{\alpha t}\right].$$

(ii) The characteristic roots $r_{1,2} = \alpha\,,\beta\,;\ \alpha = \beta\,.$ $y_1(t) = e^{\alpha t}$ and $y_2(t) = te^{\alpha t}$.

$$K(t) = t\,e^{\alpha t}.$$

(iii) The characteristic roots $r_{1,2} = \lambda \pm i\,\mu\,.$ $y_1(t) = e^{\lambda t} cos\,\mu t$ and $y_2(t) = e^{\lambda t} sin\,\mu t$.

$$K(t) = \frac{1}{\mu} e^{\lambda t} sin\,\mu t\,.$$

28. Let $y(t) = v(t)y_1(t)$, in which $y_1(t)$ is a solution of the *homogeneous equation*. Substitution into the given ODE results in

$$v''y_1 + 2v'y_1' + vy_1'' + p(t)[v'y_1 + vy_1'] + q(t)vy_1 = g(t)\,.$$

By assumption, $y_1'' + p(t)y_1 + q(t)y_1 = 0$, hence $v(t)$ must be a solution of the ODE

$$v''y_1 + [2y_1' + p(t)y_1]v' = g(t)\,.$$

Setting $w = v'$, we also have $w'y_1 + [2y_1' + p(t)y_1]w = g(t)$.

30. First write the equation as $y'' + 7t^{-1}y + 5t^{-2}\,y = t^{-1}$. As shown in Prob. 28, the function $y(t) = t^{-1}v(t)$ is a solution of the given ODE as long as v is a solution of

$$t^{-1}\,v'' + [\,-2t^{-2} + 7t^{-2}]v' = t^{-1}\,,$$

that is, $v'' + 5t^{-1}\,v' = 1$. This ODE is *linear and first order* in v'. The integrating factor is $\mu = t^5$. The solution is $v' = t/6 + c\,t^{-5}$. Direct integration now results in $v(t) = t^2/12 + c_1t^{-4} + c_2$. Hence $y(t) = t/12 + c_1t^{-5} + c_2t^{-1}$.

31. Write the equation as $y'' - t^{-1}(1+t)y + t^{-1}\,y = t\,e^{2t}$. As shown in Prob. 28, the function $y(t) = (1+t)v(t)$ is a solution of the given ODE as long as v is a solution of

$$(1+t)\,v'' + \left[2 - t^{-1}(1+t)^2\right]v' = t\,e^{2t},$$

that is, $v'' - \frac{1+t^2}{t(t+1)}\,v' = \frac{t}{t+1}e^{2t}$. This equation is first order linear in v', with integrating factor $\mu = t^{-1}(1+t)^2e^{-t}$. The solution is $v' = (t^2e^{2t} + c_1te^t)/(1+t)^2$. Integrating, we obtain $v(t) = e^{2t}/2 - e^{2t}/(t+1) + c_1e^t/(t+1) + c_2$. Hence the solution of the original ODE is $y(t) = (t-1)e^{2t}/2 + c_1e^t + c_2(t+1)$.

32. Write the equation as $y'' + t(1-t)^{-1}y - (1-t)^{-1}\,y = 2(1-t)\,e^{-t}$. The function $y(t) = e^tv(t)$ is a solution to the given ODE as long as v is a solution of

$$e^t v'' + \left[2e^t + t(1-t)^{-1}e^t\right]v' = 2(1-t)\,e^{-t},$$

that is, $v'' + [(2-t)/(1-t)]v' = 2(1-t)\,e^{-2t}$. This equation is first order linear in v', with integrating factor $\mu = e^t/(t-1)$. The solution is

$$v' = (t-1)\left(2e^{-2t} + c_1 e^{-t}\right).$$

Integrating, we obtain $v(t) = (1/2 - t)e^{-2t} - c_1 t e^{-t} + c_2$. Hence the solution of the original ODE is $y(t) = (1/2 - t)e^{-t} - c_1 t + c_2 e^t$.

Section 3.8

1. $R\cos\delta = 3$ and $R\sin\delta = 4 \Rightarrow R = \sqrt{25} = 5$ and $\delta = arctan(4/3)$. Hence

$$u = 5\cos(2t - 0.9273).$$

3. $R\cos\delta = 4$ and $R\sin\delta = -2 \Rightarrow R = \sqrt{20} = 2\sqrt{5}$ and $\delta = -arctan(1/2)$. Hence

$$u = 2\sqrt{5}\cos(3t + 0.4636).$$

4. $R\cos\delta = -2$ and $R\sin\delta = -3 \Rightarrow R = \sqrt{13}$ and $\delta = \pi + arctan(3/2)$. Hence

$$u = \sqrt{13}\cos(\pi t - 4.1244).$$

5. The spring constant is $k = 2/(1/2) = 4$ *lb/ft*. Mass $m = 2/32 = 1/16$ *lb-s²/ft*. Since there is no damping, the equation of motion is

$$\frac{1}{16}u'' + 4u = 0\,,$$

that is, $u'' + 64u = 0$. The initial conditions are $u(0) = 1/4\,ft$, $u'(0) = 0\,fps$. The general solution is $u(t) = A\cos 8t + B\sin 8t$. Invoking the initial conditions, we have $u(t) = \frac{1}{4}\cos 8t$. $R = 3$ *inches*, $\delta = 0$ *rad*, $\omega_0 = 8$ *rad/s*, and $T = \pi/4$ *sec*.

7. The spring constant is $k = 3/(1/4) = 12$ *lb/ft*. Mass $m = 3/32$ *lb-s²/ft*. Since there is no damping, the equation of motion is

$$\frac{3}{32}u'' + 12u = 0\,,$$

that is, $u'' + 128u = 0$. The initial conditions are $u(0) = -1/12\,ft$, $u'(0) = 2\,fps$. The general solution is $u(t) = A\cos 8\sqrt{2}\,t + B\sin 8\sqrt{2}\,t$. Invoking the initial conditions, we have

$$u(t) = -\frac{1}{12}\,cos\,8\sqrt{2}\,t + \frac{1}{4\sqrt{2}}\,sin\,8\sqrt{2}\,t\,.$$

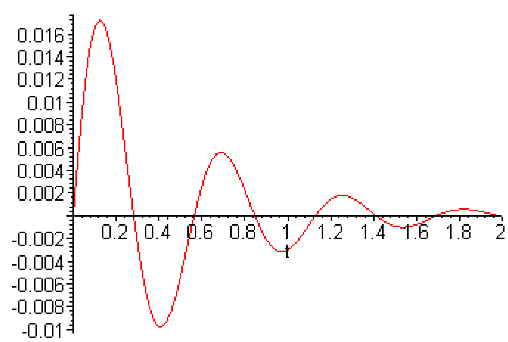
$R = \sqrt{11}/12\,ft$, $\delta = \pi - atan\left(3/\sqrt{2}\right)\,rad$, $\omega_0 = 8\sqrt{2}\,rad/s$, and $T = \pi/\left(4\sqrt{2}\right)\,sec$.

10. The spring constant is $k = 16/(1/4) = 64\,lb/ft$. Mass $m = 1/2\,lb\text{-}s^2/ft$. The damping coefficient is $\gamma = 2\,lb\text{-}sec/ft$. Hence the equation of motion is

$$\frac{1}{2}u'' + 2u' + 64u = 0\,,$$

that is, $u'' + 4u' + 128u = 0$. The initial conditions are $u(0) = 0\,ft$, $u'(0) = 1/4\,fps$. The general solution is $u(t) = A\,cos\,2\sqrt{31}\,t + B\,sin\,2\sqrt{31}\,t$. Invoking the initial conditions, we have

$$u(t) = \frac{1}{8\sqrt{31}}e^{-2t}\,sin\,2\sqrt{31}\,t\,.$$



Solving $u(t) = 0$, on the interval $[0.2\,,\,0.4]$, we obtain $t = \pi/2\sqrt{31} = 0.2821\,sec$. Based on the graph, and the solution of $u(t) = 0.01$, we have $|u(t)| \leq 0.01$ for $t \geq \tau = 0.2145$.

11. The spring constant is $k = 3/(.1) = 30\,N/m$. The damping coefficient is given as $\gamma = 3/5\,N\text{-}sec/m$. Hence the equation of motion is

$$2u'' + \frac{3}{5}u' + 30u = 0\,,$$

that is, $u'' + 0.3u' + 15u = 0$. The initial conditions are $u(0) = 0.05\,m$ and $u'(0) = 0.01\,m/s$. The general solution is $u(t) = A\,cos\,\mu t + B\,sin\,\mu t$, in which $\mu = 3.87008\,rad/s$. Invoking the initial conditions, we have

$$u(t) = e^{-0.15t}(0.05cos\,\mu t + 0.00452sin\,\mu t)\,.$$

Also, $\mu/\omega_0 = 3.87008/\sqrt{15} \approx 0.99925$.

13. The frequency of the *undamped* motion is $\omega_0 = 1$. The quasi frequency of the damped
motion is $\mu = \frac{1}{2}\sqrt{4-\gamma^2}$. Setting $\mu = \frac{2}{3}\omega_0$, we obtain $\gamma = \frac{2}{3}\sqrt{5}$.

14. The spring constant is $k = mg/L$. The equation of motion for an undamped system is

$$mu'' + \frac{mg}{L}u = 0\,.$$

Hence the natural frequency of the system is $\omega_0 = \sqrt{\frac{g}{L}}$. The period is $T = 2\pi/\omega_0$.

15. The general solution of the system is $u(t) = A\,cos\,\gamma(t-t_0) + B\,sin\,\gamma(t-t_0)$. Invoking the initial conditions, we have $u(t) = u_0 cos\gamma(t-t_0) + (u_0'/\gamma)sin\gamma(t-t_0)$. Clearly, the functions $v = u_0 cos\gamma(t-t_0)$ and $w = (u_0'/\gamma)sin\gamma(t-t_0)$ satisfy the given criteria.

16. Note that $r\,sin(\,\omega_0 t - \theta) = r\,sin\omega_0 t\,cos\,\theta - r\,cos\omega_0 t\,sin\theta$. Comparing the given expressions, we have $A = -r\,sin\,\theta$ and $B = r\,cos\,\theta$. That is, $r = R = \sqrt{A^2+B^2}$, and $tan\,\theta = -A/B = -1/tan\,\delta$. The latter relation is also $tan\,\theta + cot\,\delta = 1$.

18. The system is *critically damped*, when $R = 2\sqrt{L/C}$. Here $R = 1000$ *ohms*.

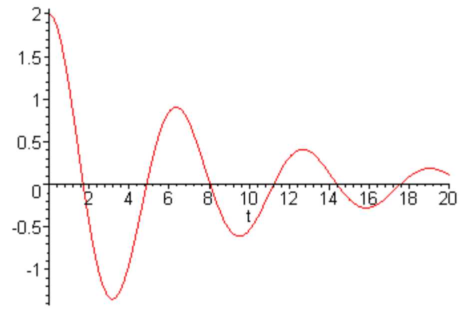
$21(a)$. Let $u = Re^{-\gamma t/2m}cos(\mu t - \delta)$. Then attains a *maximum* when $\mu t_k - \delta = 2k\pi$. Hence $T_d = t_{k+1} - t_k = 2\pi/\mu$.

(b). $u(t_k)/u(t_{k+1}) = exp(-\gamma t_k/2m)/exp(-\gamma t_{k+1}/2m) = exp[(\gamma t_{k+1} - \gamma t_k)/2m]$. Hence $u(t_k)/u(t_{k+1}) = exp[\gamma(2\pi/\mu)/2m] = exp(\gamma T_d/2m)$.

(c). $\Delta = ln[u(t_k)/u(t_{k+1})] = \gamma(2\pi/\mu)/2m = \pi\gamma/\mu m$.

22. The spring constant is $k = 16/(1/4) = 64$ *lb/ft*. Mass $m = 1/2$ *lb-s²/ft*. The damping coefficient is $\gamma = 2$ *lb-sec/ft*. The quasi frequency is $\mu = 2\sqrt{31}$ *rad/s*. Hence $\Delta = \frac{2\pi}{\sqrt{31}} \approx 1.1285$.

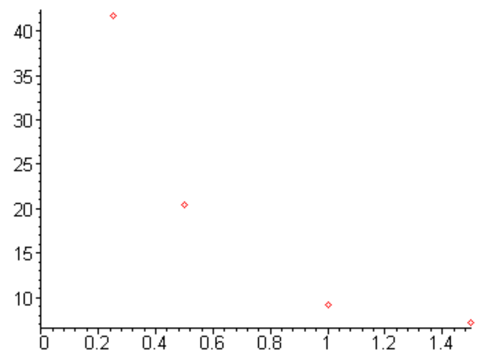
$25(a)$. The solution of the IVP is $u(t) = e^{-t/8}\left(2\,cos\,\frac{3}{8}\sqrt{7}\,t + 0.252 sin\,\frac{3}{8}\sqrt{7}\,t\right)$.



Using the plot, and numerical analysis, $\tau \approx 41.715$.

(b). For $\gamma = 0.5$, $\tau \approx 20.402$; for $\gamma = 1.0$, $\tau \approx 9.168$; for $\gamma = 1.5$, $\tau \approx 7.184$.

(c).



(d). For $\gamma = 1.6$, $\tau \approx 7.218$; for $\gamma = 1.7$, $\tau \approx 6.767$; for $\gamma = 1.8$, $\tau \approx 5.473$; for $\gamma = 1.9$, $\tau \approx 6.460$. τ steadily decreases to about $\tau_{min} \approx 4.873$, corresponding to the critical value $\gamma_0 \approx 1.73$.

(e). We have $u(t) = \frac{4e^{-\gamma t/2}}{\sqrt{4-\gamma^2}} cos(\mu t - \delta)$, in which $\mu = \frac{1}{2}\sqrt{4-\gamma^2}$, and $\delta = tan^{-1}\frac{\gamma}{\sqrt{4-\gamma^2}}$. Hence $|u(t)| \leq \frac{4e^{-\gamma t/2}}{\sqrt{4-\gamma^2}}$.

26(a). The characteristic equation is $mr^2 + \gamma r + k = 0$. Since $\gamma^2 < 4km$, the roots are $r_{1,2} = -\frac{\gamma}{2m} \pm i\frac{\sqrt{4mk-\gamma^2}}{2m}$. The general solution is

$$u(t) = e^{-\gamma t/2m}\left[A\,cos\frac{\sqrt{4mk-\gamma^2}}{2m}t + B\,sin\frac{\sqrt{4mk-\gamma^2}}{2m}t\right].$$

Invoking the initial conditions, $A = u_0$ and

$$B = \frac{(2mv_0 - \gamma u_0)}{\sqrt{4mk-\gamma^2}}.$$

(b). We can write $u(t) = R\,e^{-\gamma t/2m} cos(\mu t - \delta)$, in which

$$R = \sqrt{u_0^2 + \frac{(2mv_0 - \gamma u_0)^2}{4mk - \gamma^2}}\,,$$

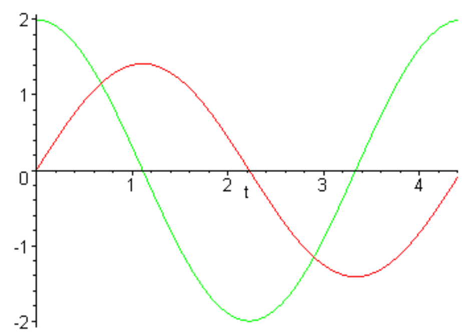
and

$$\delta = arctan\left[\frac{(2mv_0 - \gamma u_0)}{u_0\sqrt{4mk - \gamma^2}}\right].$$

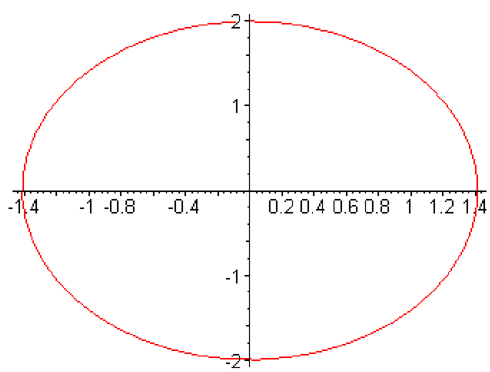
(c). $R = \sqrt{u_0^2 + \frac{(2mv_0 - \gamma u_0)^2}{4mk - \gamma^2}} = 2\sqrt{\frac{m(ku_0^2 + \gamma u_0 v_0 + mv_0^2)}{4mk - \gamma^2}} = \sqrt{\frac{a + b\gamma}{4mk - \gamma^2}}$.
It is evident that R increases (*monotonically*) without bound as $\gamma \rightarrow \left(2\sqrt{mk}\right)^-$.

28(a). The general solution is $u(t) = Acos\,\sqrt{2}\,t + Bsin\,\sqrt{2}\,t$. Invoking the initial conditions, we have $u(t) = \sqrt{2}\ sin\,\sqrt{2}\ t$.

(b).

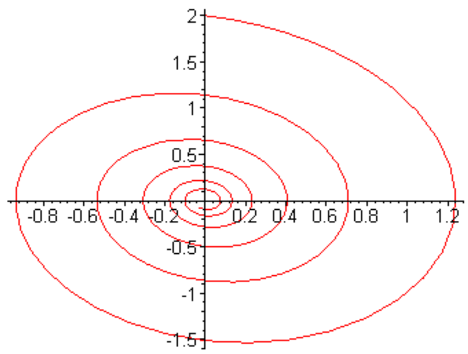


(c).



The condition $u'(0) = 2$ implies that $u(t)$ *initially* increases. Hence the phase point travels *clockwise*.

29. $u(t) = \frac{16}{\sqrt{127}} e^{-t/8} sin\, \frac{\sqrt{127}}{8} t\,.$



31. Based on *Newton's second law*, with the positive direction to the right,

$$\sum F = mu''$$

where

$$\sum F = -ku - \gamma u'.$$

Hence the equation of motion is $mu'' + \gamma u' + ku = 0$. The only difference in this problem is that the equilibrium position is located at the *unstretched* configuration of the spring.

$32(a)$. The *restoring* force exerted by the spring is $F_s = -(ku + \varepsilon u^3)$. The *opposing* viscous force is $F_d = -\gamma u'$. Based on *Newton's second law*, with the positive direction to the right,

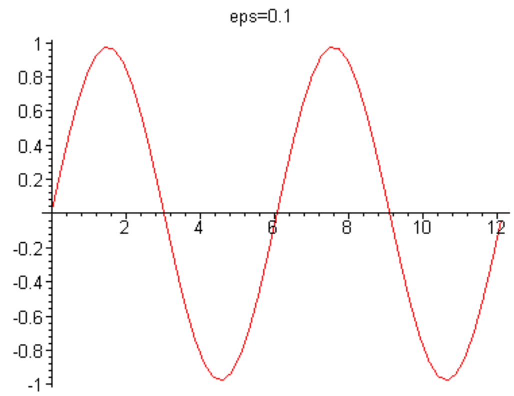
$$F_s + F_d = mu''.$$

Hence the equation of motion is $mu'' + \gamma u' + ku + \varepsilon u^3 = 0$.

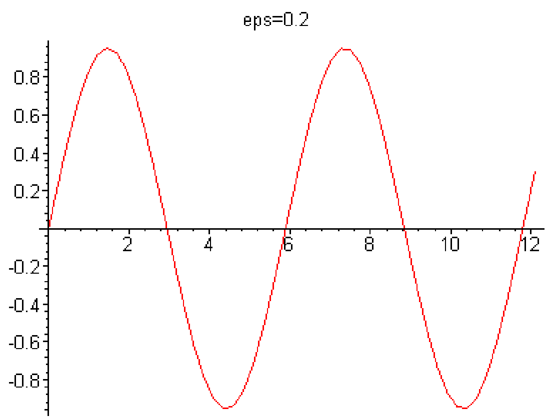
(b). With the specified parameter values, the equation of motion is $u'' + u = 0$. The general solution of this ODE is $u(t) = A\,cos\,t + B\,sin\,t$. Invoking the initial conditions,
the specific solution is $u(t) = sin\,t$. Clearly, the amplitude is $R = 1$, and the period of the motion is $T = 2\pi$.

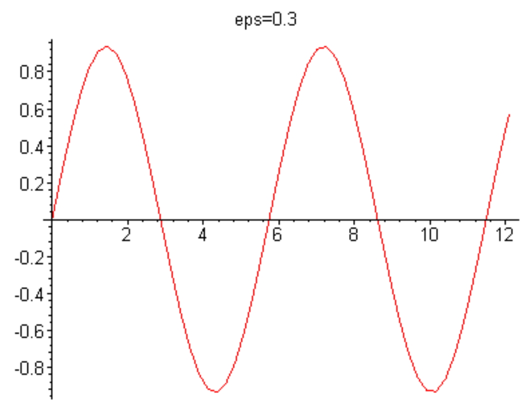
(c). Given $\varepsilon = 0.1$, the equation of motion is $u'' + u + 0.1\,u^3 = 0$. A solution of the

IVP can be generated numerically:



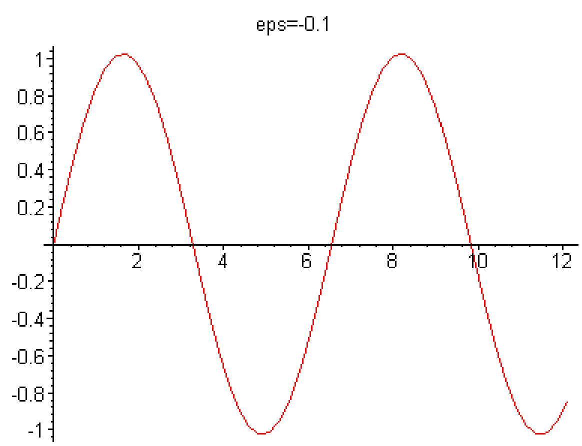
(d).

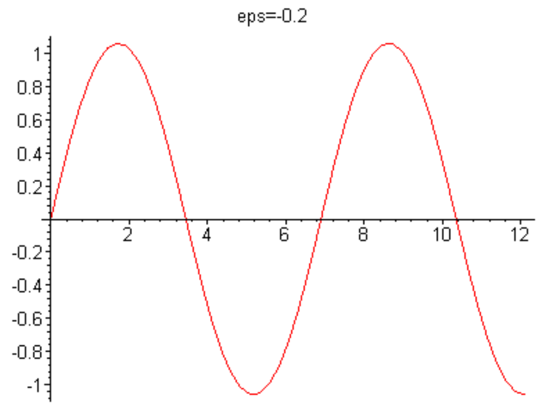




(e). The amplitude and period both seem to *decrease*.

(f).

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0.8
0.6
0.4
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-0.4
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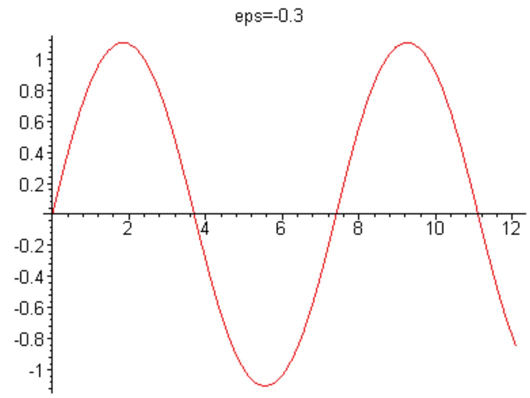
12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12

Section 3.9

2. We have $sin(\alpha\pm\beta) = sin\,\alpha\,cos\,\beta \pm cos\,\alpha\,sin\,\beta$. Subtracting the two identities, we obtain $sin(\alpha+\beta) - sin(\alpha-\beta) = 2\,cos\,\alpha\,sin\,\beta$. Setting $\alpha+\beta = 7t$ and $\alpha-\beta = 6t$, $\alpha = 6.5t$ and $\beta = 0.5t$. Hence $sin\,7t - sin\,6t = 2\,sin\,\frac{t}{2}\,cos\,\frac{13t}{2}$.

3. Consider the trigonometric identity $cos(\alpha\pm\beta) = cos\,\alpha\,cos\,\beta \mp sin\,\alpha\,sin\,\beta$. Adding the two identities, we obtain $cos(\alpha-\beta) + cos(\alpha+\beta) = 2\,cos\,\alpha\,cos\,\beta$. Comparing the expressions, set $\alpha+\beta = 2\pi t$ and $\alpha-\beta = \pi t$. Hence $\alpha = 3\pi t/2$ and $\beta = \pi t/2$. Upon substitution, we have $cos(\pi t) + cos(2\pi t) = 2\,cos(3\pi t/2)\,cos(\pi t/2)$.

4. Adding the two identities $sin(\alpha\pm\beta) = sin\,\alpha\,cos\,\beta \pm cos\,\alpha\,sin\,\beta$, it follows that $sin(\alpha-\beta) + sin(\alpha+\beta) = 2sin\,\alpha\,cos\,\beta$. Setting $\alpha+\beta = 4t$ and $\alpha-\beta = 3t$, we have $\alpha = 7t/2$ and $\beta = t/2$. Hence $sin\,3t + sin\,4t = 2\,sin(7t/2)\,cos(t/2)$.

6. Using *mks* units, the spring constant is $k = 5(9.8)/0.1 = 490$ *N/m*, and the damping coefficient is $\gamma = 2/0.04 = 50$ *N-sec/m*. The equation of motion is

$$5u'' + 50u' + 490u = 10\,sin(t/2)\,.$$

The initial conditions are $u(0) = 0$ *m* and $u'(0) = 0.03$ *m/s*.

8(a). The homogeneous solution is $u_c(t) = Ae^{-5t}cos\sqrt{73}\,t + Be^{-5t}sin\sqrt{73}\,t$. Based on the method of *undetermined coefficients*, the particular solution is

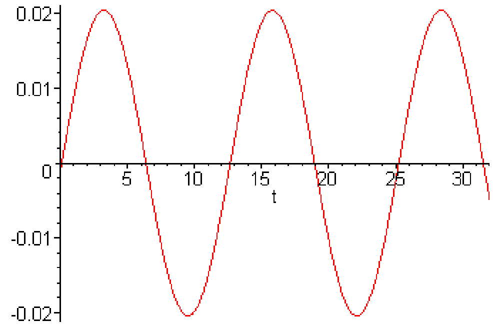
$$U(t) = \frac{1}{153281}[-160\,cos(t/2) + 3128\,sin(t/2)].$$

Hence the general solution of the ODE is $u(t) = u_c(t) + U(t)$. Invoking the initial conditions, we find that $A = 160/153281$ and $B = 383443\sqrt{73}\,/1118951300$. Hence the response is

$$u(t) = \frac{1}{153281}\left[160\,e^{-5t}cos\,\sqrt{73}\,t + \frac{383443\sqrt{73}}{7300}e^{-5t}sin\,\sqrt{73}\,t\right] + U(t).$$

(b). $u_c(t)$ is the transient part and $U(t)$ is the steady state part of the response.

(c).



(d). Based on Eqs. (9) and (10), the amplitude of the forced response is given by $R = 2/\Delta$, in which

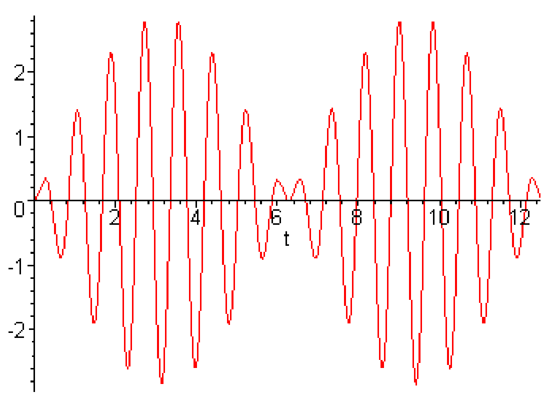
$$\Delta = \sqrt{25(98 - \omega^2)^2 + 2500\,\omega^2}\,.$$

The maximum amplitude is attained when Δ is a *minimum*. Hence the amplitude is maximum at $\omega = 4\sqrt{3}$ *rad/s* .

9. The spring constant is $k = 12$ *lb/ft* and hence the equation of motion is

$$\frac{6}{32}u'' + 12u = 4\,cos\,7t\,,$$

that is, $u'' + 64u = \frac{64}{3}cos\,7t$. The initial conditions are $u(0) = 0$ *ft*, $u'(0) = 0$ *fps*. The general solution is $u(t) = Acos\,8t + Bsin\,8t + \frac{64}{45}cos\,7t$. Invoking the initial conditions, we have $u(t) = -\frac{64}{45}cos\,8t + \frac{64}{45}cos\,7t = \frac{128}{45}sin(t/2)sin(15t/2)$.



12. The equation of motion is

$$2u'' + u' + 3u = 3cos\,3t - 2sin\,3t\,.$$

Since the system is *damped*, the steady state response is equal to the particular solution. Using the method of *undetermined coefficients*, we obtain

$$u_{ss}(t) = \frac{1}{6}(sin\,3t - cos\,3t).$$

Further, we find that $R = \sqrt{2}/6$ and $\delta = arctan(-1) = 3\pi/4$. Hence we can write $u_{ss}(t) = \frac{\sqrt{2}}{6}cos(3t - 3\pi/4)$.

13. The amplitude of the steady-state response is given by

$$R = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\,\omega^2}}\,.$$

Since F_0 is constant, the amplitude is *maximum* when the denominator of R is *minimum*. Let $z = \omega^2$, and consider the function $f(z) = m^2(\omega_0^2 - z)^2 + \gamma^2 z$. Note that $f(z)$ is a quadratic, with *minimum* at $z = \omega_0^2 - \gamma^2/2m^2$. Hence the amplitude R attains a maximum at $\omega_{max}^2 = \omega_0^2 - \gamma^2/2m^2$. Furthermore, since $\omega_0^2 = k/m$, and therefore

$$\omega_{max}^2 = \omega_0^2\left[1 - \frac{\gamma^2}{2km}\right].$$

Substituting $\omega^2 = \omega_{max}^2$ into the expression for the amplitude,

$$\begin{aligned} R &= \frac{F_0}{\sqrt{\gamma^4/4m^2 + \gamma^2\,(\omega_0^2 - \gamma^2/2m^2)}} \\ &= \frac{F_0}{\sqrt{\omega_0^2\gamma^2 - \gamma^4/4m^2}} \\ &= \frac{F_0}{\gamma\omega_0\sqrt{1 - \gamma^2/4mk}}\,. \end{aligned}$$

14(a). The forced response is $u_{ss}(t) = Acos\,\omega t + Bsin\,\omega t$. The constants are obtain by the method of *undetermined coefficients*. That is, comparing the coefficients of $cos\,\omega t$ and $sin\,\omega t$, we find that

$$-m\omega^2 A + \gamma\omega B + kA = F_0\,, \text{ and } -m\omega^2 B - \gamma\omega A + kB = 0\,.$$

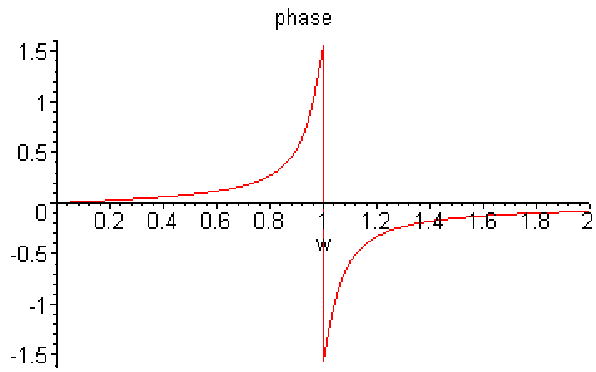
Solving this system results in

$$A = m\big(\omega_0^2 - \omega^2\big)/\Delta \quad \text{and} \quad B = \gamma\omega/\Delta\,,$$

in which $\Delta = \sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\,\omega^2}$. It follows that

$$tan\,\delta = B/A = \frac{\gamma\omega}{m(\omega_0^2 - \omega^2)}\,.$$

(b). Here $m = 1$, $\gamma = 0.125$, $\omega_0 = 1$. Hence $tan\,\delta = 0.125\omega/(1 - \omega^2)$.

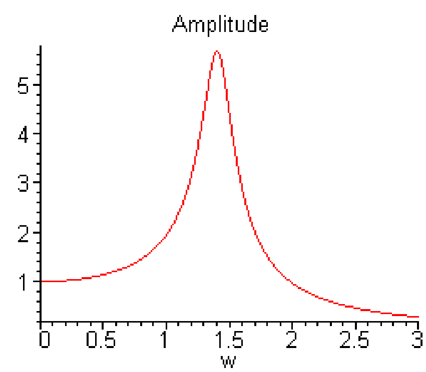


$17(a)$. Here $m = 1$, $\gamma = 0.25$, $\omega_0^2 = 2$, $F_0 = 2$. Hence $u_{ss}(t) = \frac{2}{\Delta} cos(\omega t - \delta)$, where $\Delta = \sqrt{(2-\omega^2)^2 + \omega^2/16} = \frac{1}{4}\sqrt{64 - 63\omega^2 + 16\,\omega^4}$, and $tan\,\delta = \frac{\omega}{4(2-\omega^2)}$.

(b). The amplitude is

$$R = \frac{8}{\sqrt{64 - 63\omega^2 + 16\,\omega^4}}\,.$$

(c).



(d). *See Prob. 13*. The amplitude is maximum when the denominator of R is minimum. That is, when $\omega = \omega_{max} = 3\sqrt{14}/8 \approx 1.4031$. Hence $R(\omega = \omega_{max}) = 64/\sqrt{127}$.

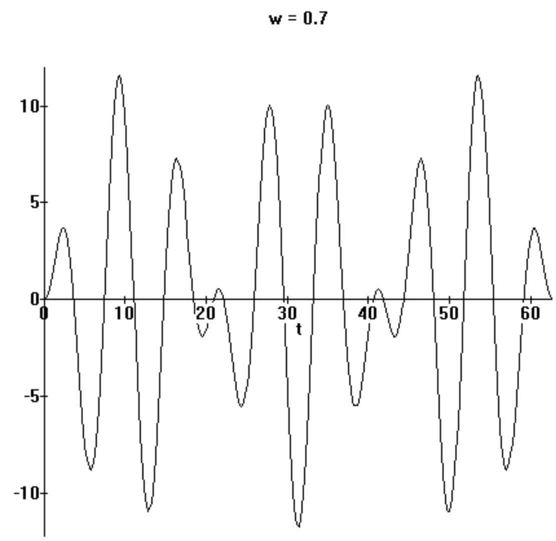
$18(a)$. The homogeneous solution is $u_c(t) = Acos\,t + Bsin\,t$. Based on the method of *undetermined coefficients*, the particular solution is

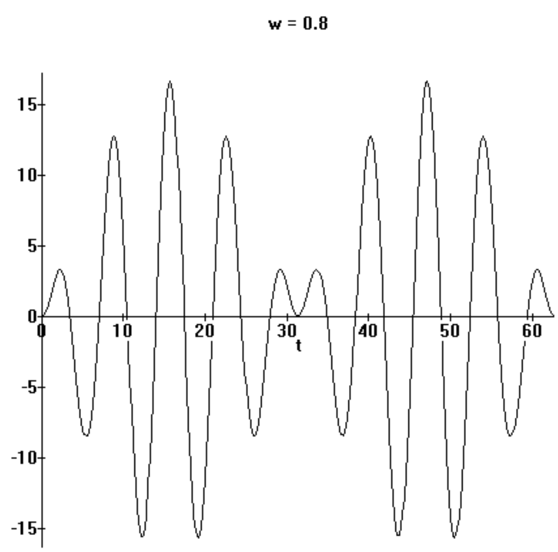
$$U(t) = \frac{3}{1-\omega^2}\,cos\,\omega t\,.$$

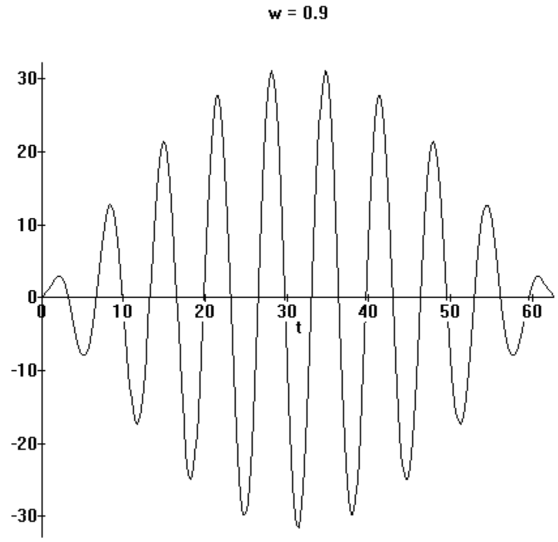
Hence the general solution of the ODE is $u(t) = u_c(t) + U(t)$. Invoking the initial conditions, we find that $A = 3/(\omega^2 - 1)$ and $B = 0$. Hence the response is

$$u(t) = \frac{3}{1-\omega^2}[\,cos\,\omega t - cos\,t\,].$$

(b).







Note that

$$u(t) = \frac{6}{1-\omega^2}\, sin\left[\frac{(1-\omega)t}{2}\right] sin\left[\frac{(\omega+1)t}{2}\right].$$

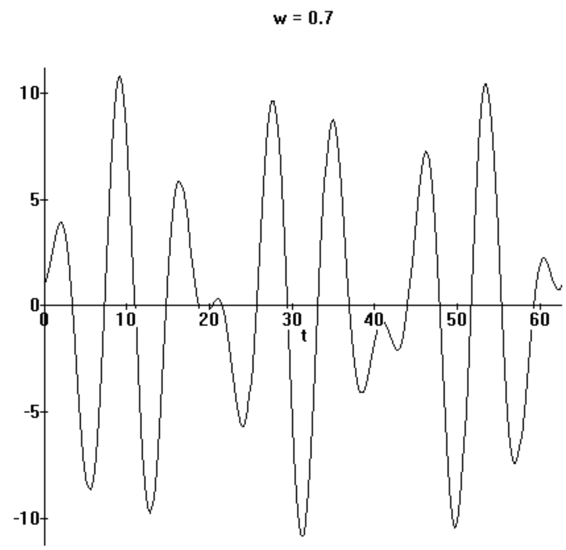
$19(a)$. The homogeneous solution is $u_c(t) = A cos\, t + B sin\, t$. Based on the method of *undetermined coefficients*, the particular solution is

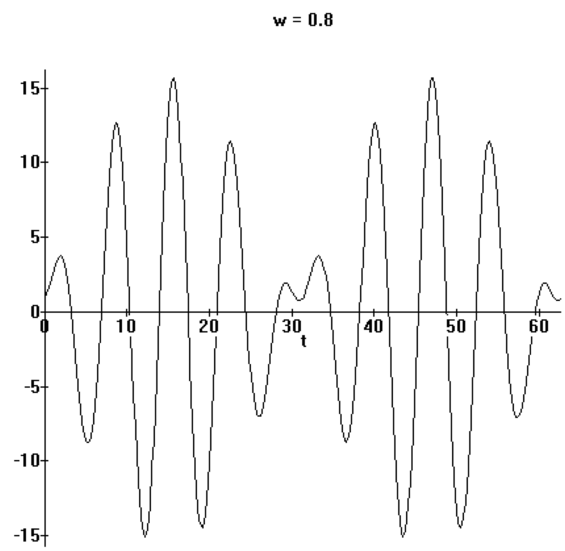
$$U(t) = \frac{3}{1-\omega^2}\, cos\, \omega t\,.$$

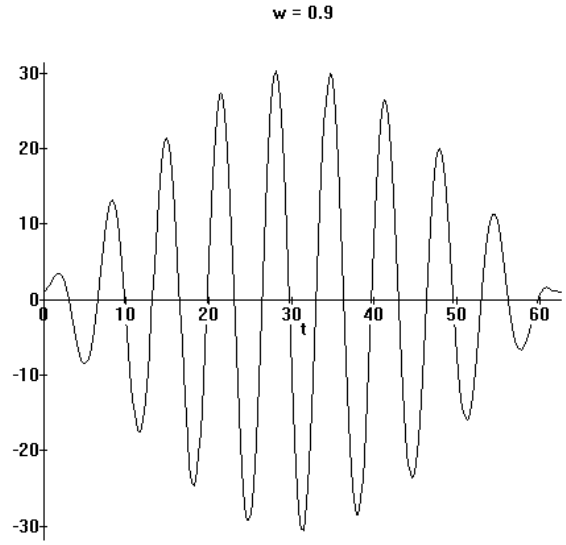
Hence the general solution is $u(t) = u_c(t) + U(t)$. Invoking the initial conditions, we find that $A = (\omega^2+2)/(\omega^2-1)$ and $B = 1$. Hence the response is

$$u(t) = \frac{1}{1-\omega^2}\left[\, 3\, cos\, \omega t - \left(\omega^2+2\right) cos\, t\,\right] + sin\, t\,.$$

$(b.)$


w = 0.7
10
5
0
-5
-10
0
10
20
30
40
50
60
t

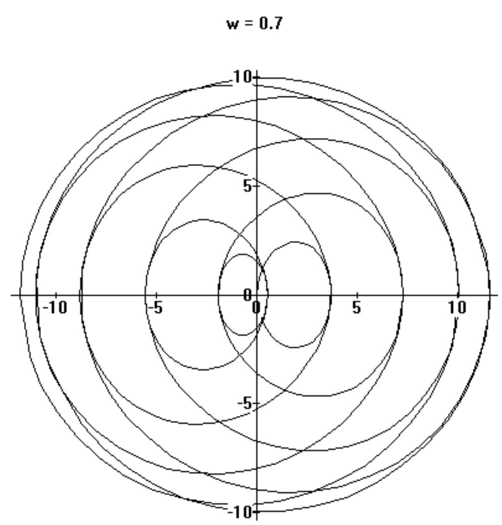

w = 0.8
15
10
5
0
-5
-10
-15
0
10
20
30
40
50
60
t

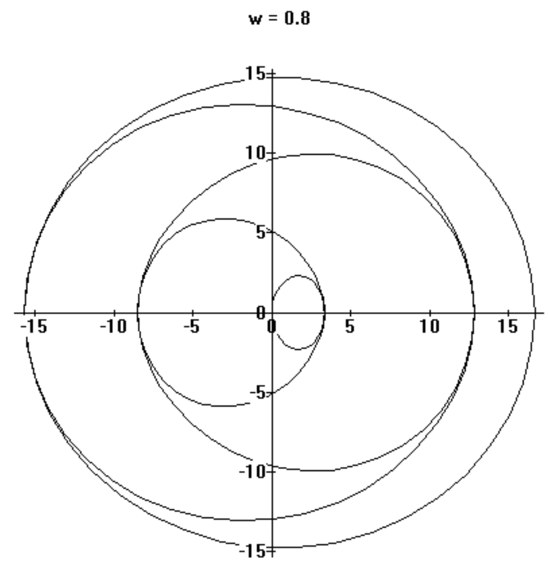


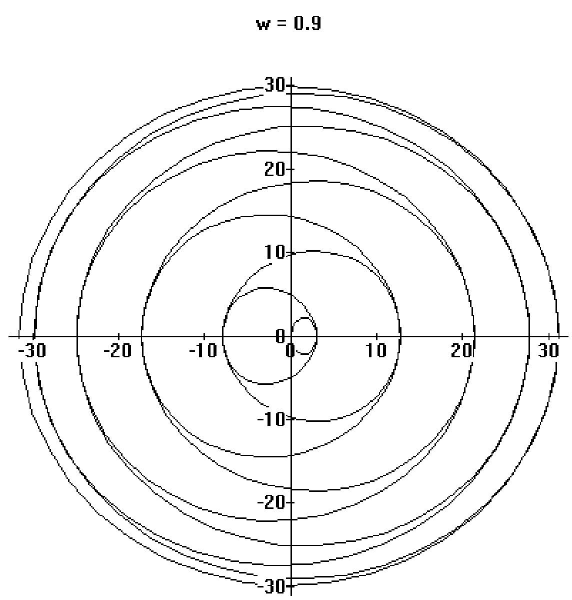
Note that

$$u(t) = \frac{6}{1-\omega^2}\, sin\left[\frac{(1-\omega)t}{2}\right] sin\left[\frac{(\omega+1)t}{2}\right] + cos\, t + sin\, t\,.$$

20.







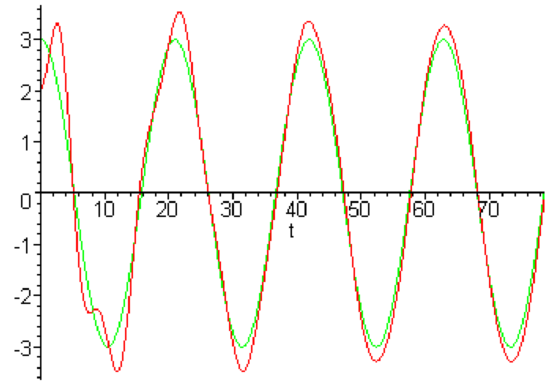
21. The general solution is $u(t) = u_c(t) + U(t)$, in which

$$u_c(t) = e^{-t/16}\left[-\frac{171358}{132721}\,cos\,\frac{\sqrt{255}}{16}t - \frac{257758}{132721\sqrt{255}}\,sin\,\frac{\sqrt{255}}{16}t\right]$$

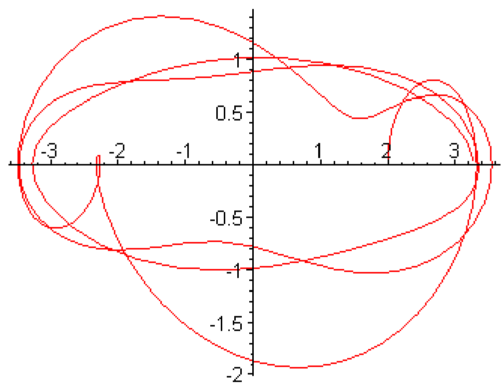
and

$$U(t) = \frac{1}{132721}\left[436800\,cos(.3t) + 18000\,sin(.3t)\right].$$

(a).



(b).



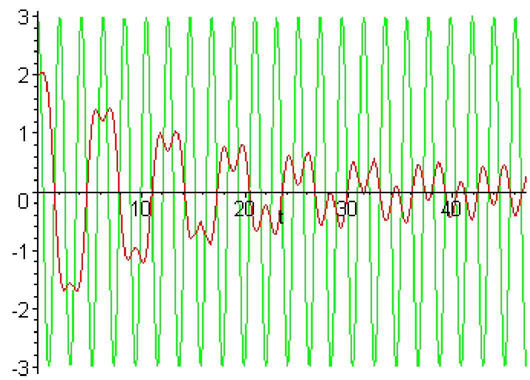
23. The general solution is $u(t) = u_c(t) + U(t)$, in which

$$u_c(t) = e^{-t/16}\left[\frac{9746}{4105}\,cos\,\frac{\sqrt{255}}{16}t + \frac{1258}{821\sqrt{255}}\,sin\,\frac{\sqrt{255}}{16}t\right]$$

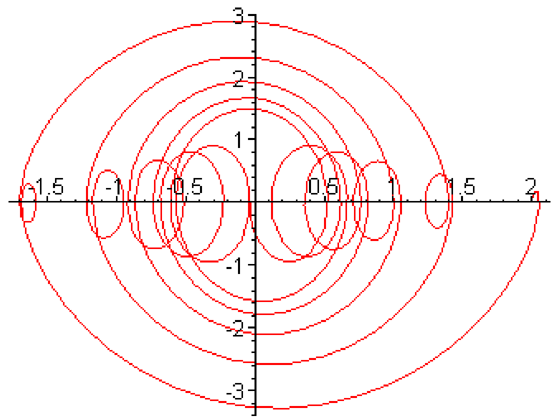
and

$$U(t) = \frac{1}{4105}\left[-1536\,cos(3t) + 72\,sin(3t)\right].$$

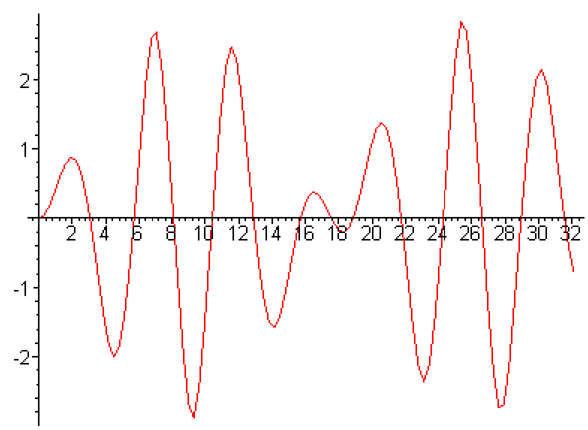
(a).


3
2
1
0
-1
-2
-3
10
20
t
30
40

(b).

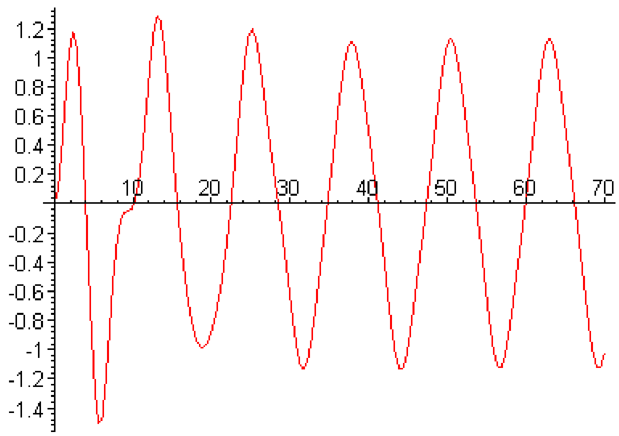

3
2
1
-1
-2
-3
-1.5
-1
-0.5
0.5
1
1.5
2

24.

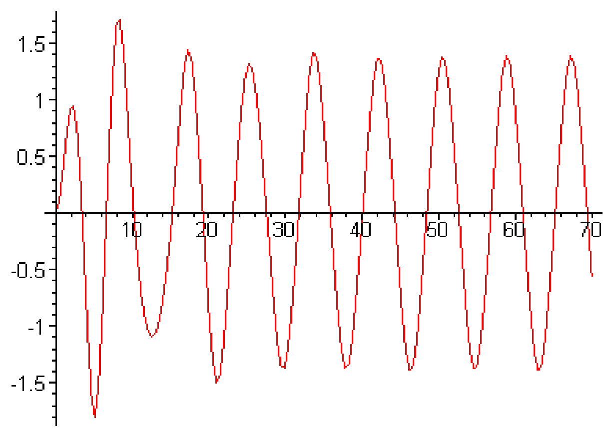

2
1
-1
-2
2
4
6
8
10
12
14
16
18
20
22
24
26
28
30
32

$25(a)$.

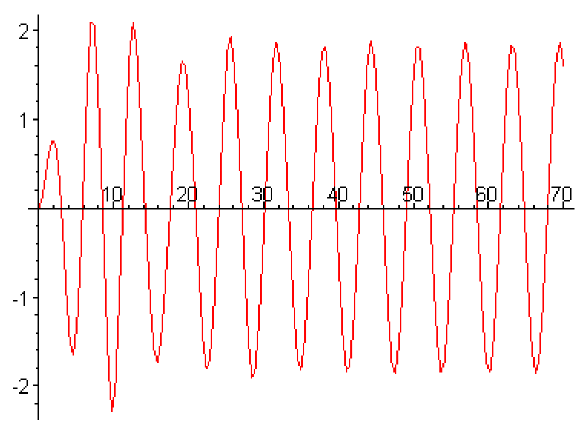
w = 0.5

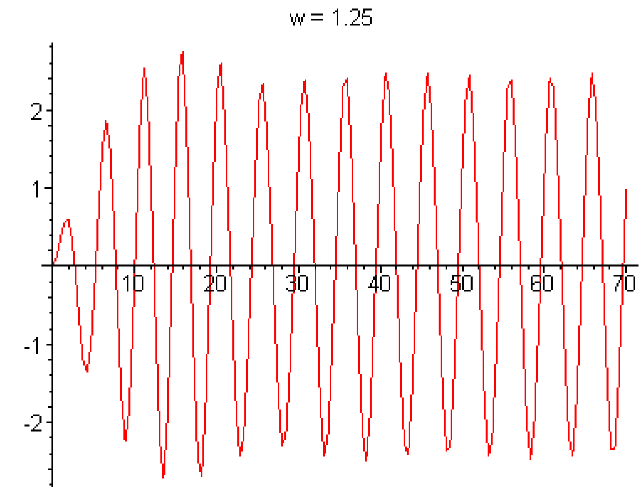

1.2
1
0.8
0.6
0.4
0.2
-0.2
-0.4
-0.6
-0.8
-1
-1.2
-1.4
10
20
30
40
50
60
70

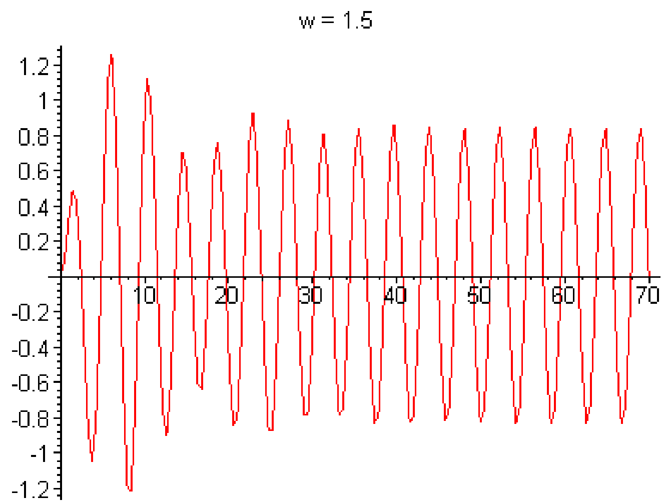
w = 0.75

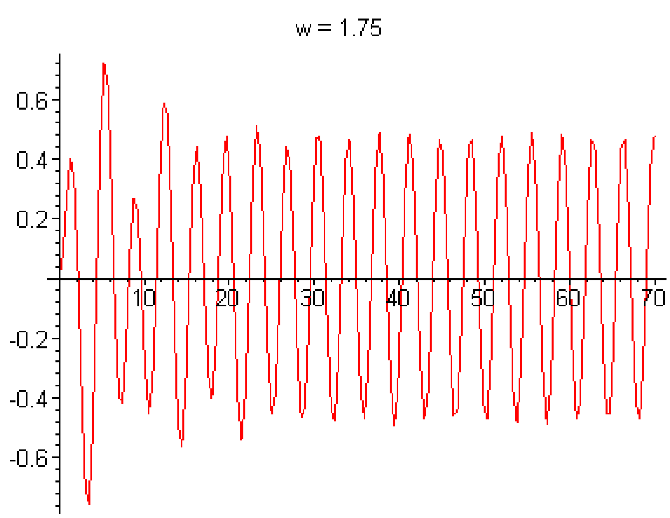

1.5
1
0.5
-0.5
-1
-1.5
10
20
30
40
50
60
70

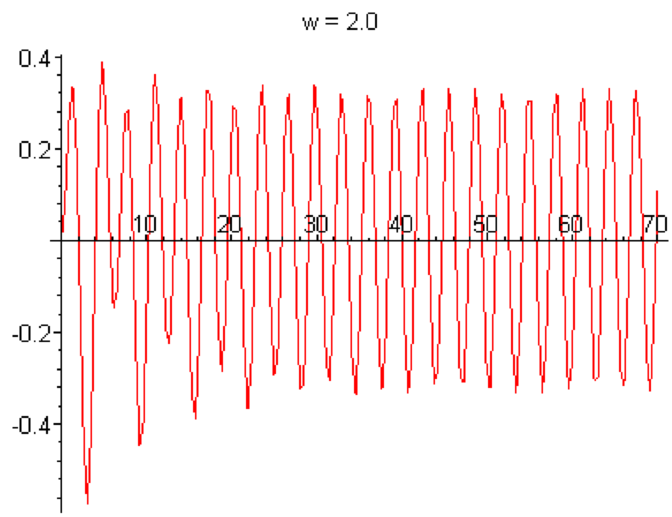
w = 1.0


2
1
-1
-2
10
20
30
40
50
60
70

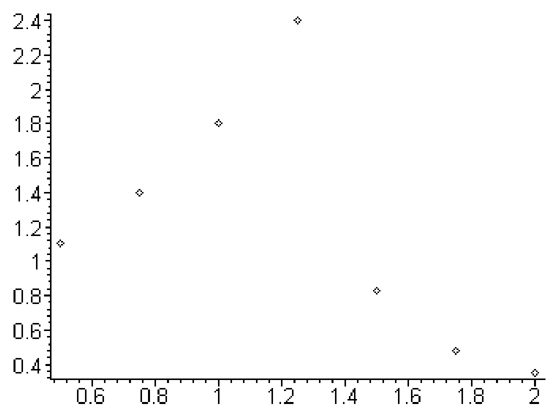

w = 1.25
2
1
-1
-2
10 20 30 40 50 60 70


w = 1.5
1.2
1
0.8
0.6
0.4
0.2
-0.2
-0.4
-0.6
-0.8
-1
-1.2
10 20 30 40 50 60 70


w = 1.75
0.6
0.4
0.2
-0.2
-0.4
-0.6
10 20 30 40 50 60 70

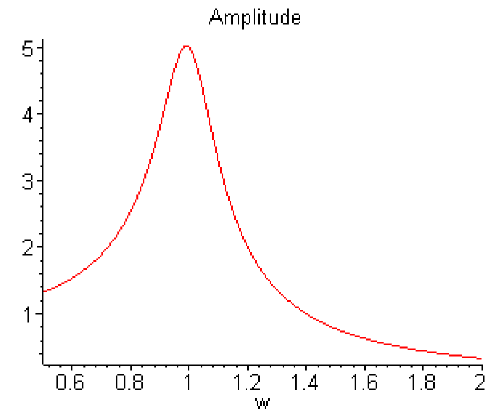


(b).



(c). The amplitude for a similar system with a *linear* spring is given by

$$R = \frac{5}{\sqrt{25 - 49\omega^2 + 25\omega^4}}\ .$$


Amplitude
5
4
3
2
1
0.6
0.8
1
1.2
1.4
1.6
1.8
2
w