Chapter Four

Section 4.1

1. The differential equation is in standard form. Its coefficients, as well as the function $g(t) = t$, are continuous *everywhere*. Hence solutions are valid on the entire real line.

3. Writing the equation in standard form, the coefficients are *rational* functions with singularities at $t = 0$ and $t = 1$. Hence the solutions are valid on the intervals $(-\infty,0)$, $(0\,,1)$, and $(1\,,\infty)$.

4. The coefficients are continuous everywhere, but the function $g(t) = ln\,t$ is defined and
continuous only on the interval $(0\,,\infty)$. Hence solutions are defined for positive reals.

5. Writing the equation in standard form, the coefficients are *rational* functions with a singularity at $x_0 = 1$. Furthermore, $p_4(x) = tan\,x/(x-1)$ is *undefined*, and hence not continuous, at $x_k = \pm(2k+1)\pi/2$, $k = 0, 1, 2, \cdots$. Hence solutions are defined on any *interval* that *does not* contain x_0 or x_k.

6. Writing the equation in standard form, the coefficients are *rational* functions with singularities at $x = \pm 2$. Hence the solutions are valid on the intervals $(-\infty, -2)$, $(-2\,,2)$, and $(2\,,\infty)$.

7. Evaluating the Wronskian of the three functions, $W(f_1, f_2, f_3) = -14$. Hence the functions are linearly *independent*.

9. Evaluating the Wronskian of the four functions, $W(f_1, f_2, f_3, f_4) = 0$. Hence the functions are linearly *dependent*. To find a linear relation among the functions, we need to find constants c_1, c_2, c_3, c_4, not all zero, such that

$$c_1 f_1(t) + c_2 f_2(t) + c_3 f_3(t) + c_4 f_4(t) = 0\,.$$

Collecting the common terms, we obtain

$$(c_2 + 2c_3 + c_4)t^2 + (2c_1 - c_3 + c_4)t + (-3c_1 + c_2 + c_4) = 0\,,$$

which results in *three* equations in *four* unknowns. Arbitrarily setting $c_4 = -1$, we can solve the equations $c_2 + 2c_3 = 1$, $2c_1 - c_3 = 1$, $-3c_1 + c_2 = 1$, to find that $c_1 = 2/7$, $c_2 = 13/7$, $c_3 = -3/7$. Hence

$$2f_1(t) + 13f_2(t) - 3f_3(t) - 7f_4(t) = 0\,.$$

10. Evaluating the Wronskian of the three functions, $W(f_1, f_2, f_3) = 156$. Hence the functions are linearly *independent*.

11. Substitution verifies that the functions are solutions of the ODE. Furthermore, we have
$W(1, cos\, t, sin\, t) = 1$.

12. Substitution verifies that the functions are solutions of the ODE. Furthermore, we have $W(1, t, cos\, t, sin\, t) = 1$.

14. Substitution verifies that the functions are solutions of the ODE. Furthermore, we have $W(1, t, e^{-t}, t\, e^{-t}) = e^{-2t}$.

15. Substitution verifies that the functions are solutions of the ODE. Furthermore, we have $W(1, x, x^3) = 6x$.

16. Substitution verifies that the functions are solutions of the ODE. Furthermore, we have $W(x, x^2, 1/x) = 6/x$.

18. The operation of taking a derivative is linear, and hence

$$(c_1 y_1 + c_2 y_2)^{(k)} = c_1 y_1^{(k)} + c_2 y_2^{(k)}.$$

It follows that

$$L[c_1 y_1 + c_2 y_2] = c_1 y_1^{(n)} + c_2 y_2^{(n)} + p_1\left[c_1 y_1^{(n-1)} + c_2 y_2^{(n-1)}\right] + \cdots + p_n[c_1 y_1 + c_2 y_2].$$

Rearranging the terms, we obtain $L[c_1 y_1 + c_2 y_2] = c_1 L[y_1] + c_2 L[y_2]$. Since y_1 and y_2 are solutions, $L[c_1 y_1 + c_2 y_2] = 0$. The rest follows by induction.

19(a). Note that $d^k(t^n)/dt^k = n(n-1)\cdots(n-k+1)t^{n-k}$, for $k = 1, 2, \cdots, n$. Hence

$$L[t^n] = a_0\, n! + a_1[n(n-1)\cdots 2]t + \cdots a_{n-1}\, n\, t^{n-1} + a_n\, t^n.$$

(b). We have $d^k(e^{rt})/dt^k = r^k e^{rt}$, for $k = 0, 1, 2, \cdots$. Hence

$$\begin{aligned} L\left[e^{rt}\right] &= a_0\, r^n e^{rt} + a_1 r^{n-1} e^{rt} + \cdots a_{n-1} r\, e^{rt} + a_n\, e^{rt} \\ &= \left[a_0\, r^n + a_1 r^{n-1} + \cdots a_{n-1}\, r\, + a_n\right] e^{rt}. \end{aligned}$$

(c). Set $y = e^{rt}$, and substitute into the ODE. It follows that $r^4 - 5r^2 + 4 = 0$, with $r = \pm 1, \pm 2$. Furthermore, $W(e^t, e^{-t}, e^{2t}, e^{-2t}) = 72$.

20(a). Let $f(t)$ and $g(t)$ be arbitrary functions. Then $W(f, g) = fg' - f'g$. Hence $W'(f, g) = f'g' + fg'' - f''g - f'g' = fg'' - f''g$. That is,

$$W'(f, g) = \begin{vmatrix} f & g \\ f'' & g'' \end{vmatrix}.$$

Now expand the 3-*by*-3 determinant as

$$W(y_1, y_2, y_3) = y_1\begin{vmatrix} y_2' & y_3' \\ y_2'' & y_3'' \end{vmatrix} - y_2\begin{vmatrix} y_1' & y_3' \\ y_1'' & y_3'' \end{vmatrix} + y_3\begin{vmatrix} y_1' & y_2' \\ y_1'' & y_2'' \end{vmatrix}.$$

Differentiating, we obtain

$$\begin{aligned} W'(y_1, y_2, y_3) &= y_1'\begin{vmatrix} y_2' & y_3' \\ y_2'' & y_3'' \end{vmatrix} - y_2'\begin{vmatrix} y_1' & y_3' \\ y_1'' & y_3'' \end{vmatrix} + y_3'\begin{vmatrix} y_1' & y_2' \\ y_1'' & y_2'' \end{vmatrix} + \\ &+ y_1\begin{vmatrix} y_2' & y_3' \\ y_2''' & y_3''' \end{vmatrix} - y_2\begin{vmatrix} y_1' & y_3' \\ y_1''' & y_3''' \end{vmatrix} + y_3\begin{vmatrix} y_1' & y_2' \\ y_1''' & y_2''' \end{vmatrix}. \end{aligned}$$

The *second* line follows from the observation above. Now we find that

$$W'(y_1, y_2, y_3) = \begin{vmatrix} y_1' & y_2' & y_3' \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} + \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1''' & y_2''' & y_3''' \end{vmatrix}.$$

Hence the assertion is true, since the first determinant is equal to *zero*.

(b). Based on the properties of determinants,

$$p_2(t)p_3(t)W' = \begin{vmatrix} p_3\, y_1 & p_3\, y_2 & p_3\, y_3 \\ p_2\, y_1' & p_2\, y_2' & p_2\, y_3' \\ y_1''' & y_2''' & y_3''' \end{vmatrix}$$

Adding the *first two* rows to the *third* row does not change the value of the determinant. Since the functions are assumed to be solutions of the given ODE, addition of the rows results in

$$p_2(t)p_3(t)W' = \begin{vmatrix} p_3\, y_1 & p_3\, y_2 & p_3\, y_3 \\ p_2\, y_1' & p_2\, y_2' & p_2\, y_3' \\ -p_1\, y_1'' & -p_1\, y_2'' & -p_1\, y_3'' \end{vmatrix}.$$

It follows that $p_2(t)p_3(t)W' = -p_1(t)p_2(t)p_3(t)W$. As long as the coefficients are not zero, we obtain $W' = -p_1(t)W$.

(c). The first order equation $W' = -p_1(t)W$ is linear, with integrating factor $\mu(t) =$ $= exp\big(\int p_1(t)dt\big)$. Hence $W(t) = c\, exp\big(-\int p_1(t)dt\big)$. Furthermore, $W(t)$ is *zero* only if $c = 0$.

(d). It can be shown, by mathematical induction, that

$$W'(y_1, y_2, \cdots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_{n-1} & y_n \\ y_1' & y_2' & \cdots & y_{n-1}' & y_n' \\ \vdots & & & & \vdots \\ y_1^{(n-2)} & y_2^{(n-2)} & \cdots & y_{n-1}^{(n-2)} & y_n^{(n-2)} \\ y_1^{(n)} & y_2^{(n)} & \cdots & y_{n-1}^{(n)} & y_n^{(n)} \end{vmatrix}.$$

Based on the reasoning in Part(b), it follows that

$$p_2(t)p_3(t)\cdots p_n(t)W' = -p_1(t)p_2(t)p_3(t)\cdots p_n(t)W,$$

and hence $W' = -p_1(t)W$.

22. Inspection of the coefficients reveals that $p_1(t) = 0$. Based on Prob. 20, we find that $W' = 0$, and hence $W = c$.

23. After writing the equation in standard form, observe that $p_1(t) = 2/t$. Based on the results in Prob. 20, we find that $W' = (-2/t)W$, and hence $W = c/t^2$.

24. Writing the equation in standard form, we find that $p_1(t) = 1/t$. Using *Abel's formula*, the Wronskian has the form $W(t) = c\,exp\big(-\int \frac{1}{t}dt\big) = c/t$.

25(a). Assuming that $c_1y_1(t) + c_2y_2(t) + \cdots + c_ny_n(t) = 0$, then taking the first $n-1$ derivatives of this equation results in

$$c_1y_1^{(k)}(t) + c_2y_2^{(k)}(t) + \cdots + c_ny_n^{(k)}(t) = 0$$

for $k = 0, 1, \cdots, n-1$. Setting $t = t_0$, we obtain a system of n algebraic equations with unknowns $c_1, c_2, \cdots, c_n$. The Wronskian, $W(y_1, y_2, \cdots, y_n)(t_0)$, is the determinant of the coefficient matrix. Since system of equations is homogeneous, $W(y_1, y_2, \cdots, y_n)(t_0) \neq 0$ implies that the only solution is the *trivial* solution, $c_1 = c_2 = \cdots = c_n = 0$.

(b). Suppose that $W(y_1, y_2, \cdots, y_n)(t_0) = 0$ for some t_0. Consider the system of algebraic
equations

$$c_1y_1^{(k)}(t_0) + c_2y_2^{(k)}(t_0) + \cdots + c_ny_n^{(k)}(t_0) = 0,$$

$k = 0, 1, \cdots, n-1$, with unknowns $c_1, c_2, \cdots, c_n$. Vanishing of the Wronskian, which is the determinant of the coefficient matrix, implies that there is a *nontrivial* solution of the system of homogeneous equations. That is, there exist constants $c_1, c_2, \cdots, c_n$, not all zero, which satisfy the above equations. Now let

$$y(t) = c_1y_1(t) + c_2y_2(t) + \cdots + c_ny_n(t).$$

Since the ODE is linear, $y(t)$ is also a *nonzero* solution. Based on the system of algebraic equations above, $y(t_0) = y'(t_0) = \cdots = y^{(n-1)}(t_0) = 0$. This contradicts the uniqueness of the *identically zero* solution.

26. Let $y(t) = y_1(t)v(t)$. Then $y' = y_1'v + y_1v'$, $y'' = y_1''v + 2y_1'v' + y_1v''$, and $y''' = y_1'''v + 3y_1''v' + 3y_1'v'' + y_1v'''$. Substitution into the ODE results in

$$y_1'''v + 3y_1''v' + 3y_1'v'' + y_1v''' + p_1[y_1''v + 2y_1'v' + y_1v''] + p_2[y_1'v + y_1v'] + p_3y_1v = 0.$$

Since y_1 is assumed to be a solution, all terms containing the factor $v(t)$ vanish. Hence

$$y_1 v''' + [p_1 y_1 + 3y_1']v'' + [3y_1'' + 2p_1 y_1' + p_2 y_1]v' = 0\,,$$

which is a *second order* ODE in the variable $u = v'$.

28. First write the equation in standard form:

$$y''' - 3\frac{t+2}{t(t+3)}y'' + 6\frac{t+1}{t^2(t+3)}y' - \frac{6}{t^2(t+3)}y = 0\,.$$

Let $y(t) = t^2 v(t)$. Substitution into the given ODE results in

$$t^2 v''' + 3\frac{t(t+4)}{t+3}v'' = 0\,.$$

Set $w = v''$. Then w is a solution of the first order differential equation

$$w' + 3\frac{t+4}{t(t+3)}w = 0\,.$$

This equation is *linear*, with integrating factor $\mu(t) = t^4/(t+3)$. The general solution is $w = c(t+3)/t^4$. Integrating twice, it follows that $v(t) = c_1 t^{-1} + c_1 t^{-2} + c_2 t + c_3$. Hence $y(t) = c_1 t + c_1 + c_2 t^3 + c_3 t^2$. Finally, since $y_1(t) = t^2$ and $y_2(t) = t^3$are given solutions, the *third* independent solution is $y_3(t) = c_1 t + c_1$.

Section 4.2

1. The *magnitude* of $1+i$ is $R=\sqrt{2}$ and the *polar angle* is $\pi/4$. Hence the polar form is given by $1+i=\sqrt{2}\,e^{i\pi/4}$.

3. The *magnitude* of -3 is $R=3$ and the *polar angle* is π. Hence $-3=3\,e^{i\pi}$.

4. The *magnitude* of $-i$ is $R=1$ and the *polar angle* is $3\pi/2$. Hence $-i=e^{3\pi i/2}$.

5. The *magnitude* of $\sqrt{3}-i$ is $R=2$ and the *polar angle* is $-\pi/6=11\pi/6$. Hence the polar form is given by $\sqrt{3}-i=2\,e^{11\pi i/6}$.

6. The *magnitude* of $-1-i$ is $R=\sqrt{2}$ and the *polar angle* is $5\pi/4$. Hence the polar form is given by $-1-i=\sqrt{2}\,e^{5\pi i/4}$.

7. Writing the complex number in polar form, $1=e^{2m\pi i}$, where m may be any integer. Thus $1^{1/3}=e^{2m\pi i/3}$. Setting $m=0,1,2$ successively, we obtain the three roots as $1^{1/3}=1$, $1^{1/3}=e^{2\pi i/3}$, $1^{1/3}=e^{4\pi i/3}$. Equivalently, the roots can also be written as 1, $cos(2\pi/3)+i\,sin(2\pi/3)=\frac{1}{2}\left(-1+\sqrt{3}\right)$, $cos(4\pi/3)+isin(4\pi/3)=\frac{1}{2}\left(-1+\sqrt{3}\right)$.

9. Writing the complex number in polar form, $1=e^{2m\pi i}$, where m may be any integer. Thus $1^{1/4}=e^{2m\pi i/4}$. Setting $m=0,1,2,3$ successively, we obtain the three roots as $1^{1/4}=1$, $1^{1/4}=e^{\pi i/2}$, $1^{1/4}=e^{\pi i}$, $1^{1/4}=e^{3\pi i/2}$. Equivalently, the roots can also be written as 1, $cos(\pi/2)+i\,sin(\pi/2)=i$, $cos(\pi)+isin(\pi)=-1$, $cos(3\pi/2)+$ $+isin(3\pi/2)=-i$.

10. In polar form, $2(cos\,\pi/3+i\,sin\,\pi/3)=2\,e^{i\pi/3+2m\pi}$, in which m is any integer. Thus $[2(cos\,\pi/3+i\,sin\,\pi/3)]^{1/2}=2^{1/2}\,e^{i\pi/6+m\pi}$. With $m=0$, one square root is given by $2^{1/2}\,e^{i\pi/6}=\left(\sqrt{3}+i\right)/\sqrt{2}$. With $m=1$, the other root is given by $2^{1/2}\,e^{i7\pi/6}=\left(-\sqrt{3}-i\right)/\sqrt{2}$.

11. The characteristic equation is $r^3-r^2-r+1=0$. The roots are $r=-1,1,1$. One root is *repeated*, hence the general solution is $y=c_1e^{-t}+c_2e^t+c_3te^t$.

13. The characteristic equation is $r^3-2r^2-r+2=0$, with roots $r=-1,1,2$. The roots are real and *distinct*, hence the general solution is $y=c_1e^{-t}+c_2e^t+c_3e^{2t}$.

14. The characteristic equation can be written as $r^2(r^2-4r+4)=0$. The roots are $r=0,0,2,2$. There are two repeated roots, and hence the general solution is given by $y=c_1+c_2t+c_3e^{2t}+c_4te^{2t}$.

15. The characteristic equation is $r^6+1=0$. The roots are given by $r=(-1)^{1/6}$, that is, the six *sixth roots* of -1. They are $e^{-\pi i/6+m\pi i/3}$, $m=0,1,\cdots,5$. Explicitly,

$r = \left(\sqrt{3} - i\right)/2\,,\, \left(\sqrt{3} + i\right)/2\,,\, i\,,\, -i\,,\, \left(-\sqrt{3} + i\right)/2\,,\, \left(-\sqrt{3} - i\right)/2$. Hence the general solution is given by $y = e^{\sqrt{3}t/2}[c_1 cos\,(t/2) + c_2 sin\,(t/2)] + c_3 cos\,t + c_4 sin\,t + e^{-\sqrt{3}t/2}[c_5 cos\,(t/2) + c_6 sin\,(t/2)]$.

16. The characteristic equation can be written as $(r^2 - 1)(r^2 - 4) = 0$. The roots are given by $r = \pm 1, \pm 2$. The roots are real and *distinct*, hence the general solution is $y = c_1 e^{-t} + c_2 e^t + c_3 e^{-2t} + c_4 e^{2t}$.

17. The characteristic equation can be written as $(r^2 - 1)^3 = 0$. The roots are given by $r = \pm 1$, each with *multiplicity three*. Hence the general solution is

$$y = c_1 e^{-t} + c_2 te^{-t} + c_3 t^2 e^{-t} + c_4 e^t + c_5 te^t + c_6 t^2 e^t.$$

18. The characteristic equation can be written as $r^2\left(r^4 - 1\right) = 0$. The roots are given by $r = 0, 0, \pm 1, \pm i$. The general solution is $y = c_1 + c_2 t + c_3 e^{-t} + c_4 e^t + c_5 cos\,t +$
$+ c_6 sin\,t$.

19. The characteristic equation can be written as $r\left(r^4 - 3r^3 + 3r^2 - 3r + 2\right) = 0$. Examining the coefficients, it follows that $r^4 - 3r^3 + 3r^2 - 3r + 2 = (r-1)(r-2) \times (r^2 + 1)$. Hence the roots are $r = 0, 1, 2, \pm i$. The general solution of the ODE is given by $y = c_1 + c_2 e^t + c_3 e^{2t} + c_4 cos\,t + c_5 sin\,t$.

20. The characteristic equation can be written as $r(r^3 - 8) = 0$, with roots $r = 0$, $2\,e^{2m\pi i/3}, m = 0, 1, 2$. That is, $r = 0, 2, -1 \pm i\sqrt{3}$. Hence the general solution is $y = c_1 + c_2 e^{2t} + e^{-t}\left[c_3 cos\sqrt{3}\,t + c_4 sin\sqrt{3}\,t\right]$.

21. The characteristic equation can be written as $\left(r^4 + 4\right)^2 = 0$. The roots of the equation $r^4 + 4 = 0$ are $r = 1 \pm i\,,\, -1 \pm i$. Each of these roots has $multiplicity\,two$. The general solution is $y = e^t[c_1 cos\,t + c_2 sin\,t] + te^t[c_3 cos\,t + c_4 sin\,t] +$
$+ e^{-t}[c_5 cos\,t + c_6 sin\,t] + te^{-t}[c_7 cos\,t + c_8 sin\,t]$.

22. The characteristic equation can be written as $(r^2 + 1)^2 = 0$. The roots are given by $r = \pm i$, each with $multiplicity\,two$. The general solution is $y = c_1 cos\,t + c_2 sin$
$t +$
$+ t[c_3 cos\,t + c_4 sin\,t]$.

24. The characteristic equation is $r^3 + 5r^2 + 6r + 2 = 0$. Examining the coefficients, we find that $r^3 + 5r^2 + 6r + 2 = (r+1)(r^2 + 4r + 2)$. Hence the roots are deduced as $r = -1\,,\, -2 \pm \sqrt{2}$. The general solution is $y = c_1 e^{-t} + c_2 e^{\left(-2+\sqrt{2}\right)t} + c_3 e^{\left(-2-\sqrt{2}\right)t}$.

25. The characteristic equation is $18r^3 + 21r^2 + 14r + 4 = 0$. By examining the first and last coefficients, we find that $18r^3 + 21r^2 + 14r + 4 = (2r+1)(9r^2 + 6r + 4)$.

Hence the roots are $r = -1/2\,,\ \left(-1 \pm \sqrt{3}\right)/3$. The general solution of the ODE is given by $y = c_1 e^{-t/2} + e^{-t/3}\left[c_2 cos\left(t/\sqrt{3}\right) + c_3 sin\left(t/\sqrt{3}\right)\right]$.

26. The characteristic equation is $r^4 - 7r^3 + 6r^2 + 30r - 36 = 0$. By examining the first and last coefficients, we find that

$$r^4 - 7r^3 + 6r^2 + 30r - 36 = (r-3)(r+2)\left(r^2 - 6r + 6\right).$$

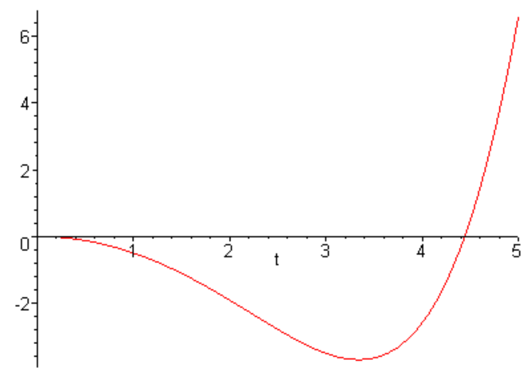
The roots are $r = -2, 3, 3 \pm \sqrt{3}$. The general solution is

$$y = c_1 e^{-2t} + c_2 e^{3t} + c_3 e^{\left(3-\sqrt{3}\right)t} + c_4 e^{\left(3+\sqrt{3}\right)t}.$$

28. The characteristic equation is $r^4 + 6r^3 + 17r^2 + 22r + 14 = 0$. It can be shown that $r^4 + 6r^3 + 17r^2 + 22r + 14 = (r^2 + 2r + 2)(r^2 + 4r + 7)$. Hence the roots are $r = -1 \pm i\,,\ -2 \pm i\sqrt{3}$. The general solution is

$$y = e^{-t}[c_1 cos\, t + c_2 sin\, t] + e^{-2t}\left[c_3 cos\sqrt{3}\,t + c_4 sin\sqrt{3}\,t\right].$$

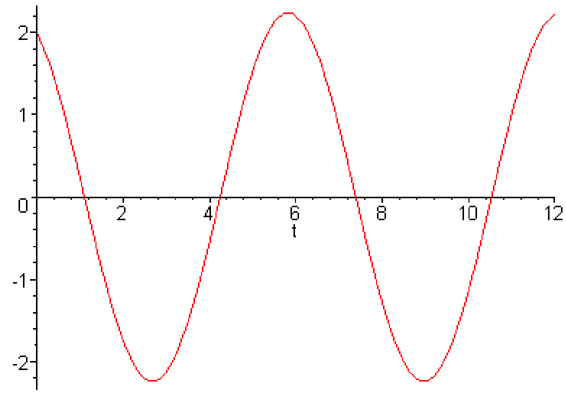
30. $y(t) = \frac{1}{2}e^{-t/\sqrt{2}} sin\left(t/\sqrt{2}\right) - \frac{1}{2}e^{t/\sqrt{2}} sin\left(t/\sqrt{2}\right)$.



32. The characteristic equation is $r^3 - r^2 + r - 1 = 0$, with roots $r = 1\,,\pm i$. Hence the general solution is $y(t) = c_1 e^t + c_2 cos\, t + c_3 sin\, t$. Invoking the initial conditions, we obtain the system of equations

$$\begin{aligned} c_1 + c_2 &= 2 \\ c_1 + c_3 &= -1 \\ c_1 - c_2 &= -2 \end{aligned}$$

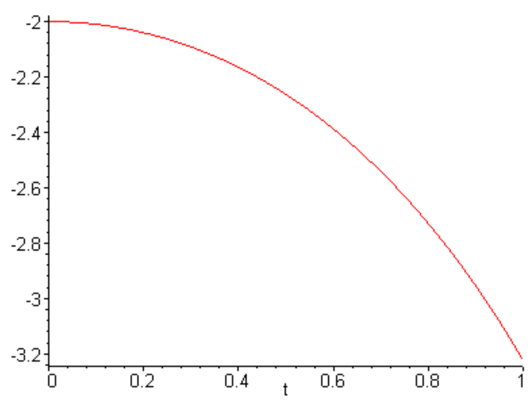
with solution $c_1 = 0\,, c_2 = 2\,, c_3 = -1$. Therefore the solution of the initial value problem is $y(t) = 2cos\, t - sin\, t$.



33. The characteristic equation is $2r^4 - r^3 - 9r^2 + 4r + 4 = 0$, with roots $r = -1/2$, $1, \pm 2$. Hence the general solution is $y(t) = c_1 e^{-t/2} + c_2 e^t + c_3 e^{-2t} + c_4 e^{2t}$. Applying the initial conditions, we obtain the system of equations

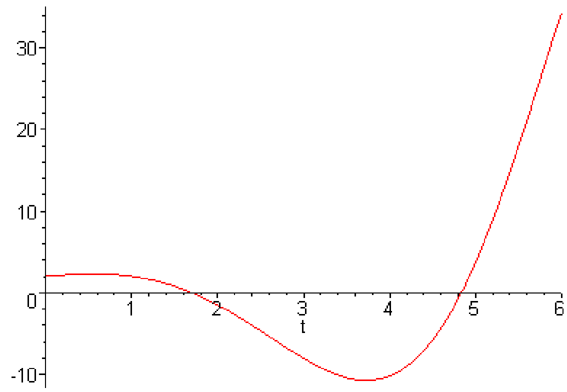
$$\begin{aligned} c_1 + c_2 + c_3 + c_4 &= -2 \\ -\frac{1}{2}c_1 + c_2 - 2c_3 + 2c_4 &= 0 \\ \frac{1}{4}c_1 + c_2 + 4c_3 + 4c_4 &= -2 \\ -\frac{1}{8}c_1 + c_2 - 8c_3 + 8c_4 &= 0 \end{aligned}$$

with solution $c_1 = -16/15$, $c_2 = -2/3$, $c_3 = -1/6$, $c_4 = -1/10$. Therefore the solution of the initial value problem is $y(t) = -\frac{16}{15}e^{-t/2} - \frac{2}{3}e^t - \frac{1}{6}e^{-2t} - \frac{1}{10}e^{2t}$.



The solution decreases without bound.

34. $y(t) = \frac{2}{13}e^{-t} + e^{t/2}\left[\frac{24}{13}cos\, t + \frac{3}{13}sin\, t\right]$.

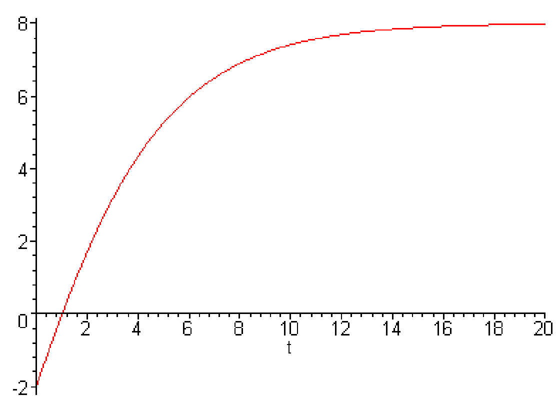


The solution is an oscillation with *increasing* amplitude.

35. The characteristic equation is $6\,r^3+5r^2+r=0$, with roots $r=0\,,-1/3\,,\,-1/2$. The general solution is $y(t)=c_1+c_2e^{-t/3}+c_3e^{-t/2}$. Invoking the initial conditions, we require that

$$\begin{aligned} c_1+c_2+c_3 &= -2 \\ -\frac{1}{3}c_2-\frac{1}{2}c_3 &= 2 \\ \frac{1}{9}c_2+\frac{1}{4}c_3 &= 0 \end{aligned}$$

with solution $c_1=8\,,\,c_2=-18\,,\,c_3=8$. Therefore the solution of the initial value problem is $y(t)=8-18e^{-t/3}+8e^{-t/2}$.



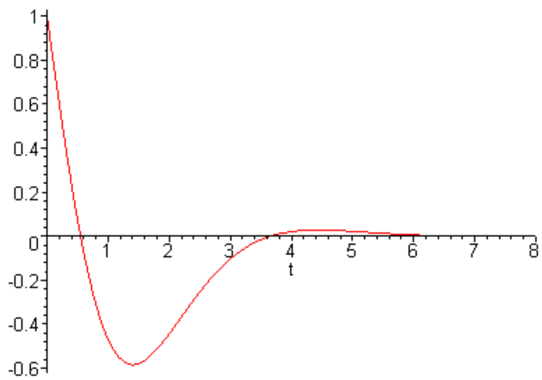
36. The general solution is derived in Prob.(28) as

$$y(t)=e^{-t}[c_1cos\,t+c_2sin\,t]+e^{-2t}\left[c_3cos\sqrt{3}\,t+c_4sin\sqrt{3}\,t\right].$$

Invoking the initial conditions, we obtain the system of equations

$$
\begin{aligned}
c_1 + c_3 &= 1 \\
-c_1 + c_2 - 2c_3 + \sqrt{3}\,c_4 &= -2 \\
-2c_2 + c_3 - 4\sqrt{3}\,c_4 &= 0 \\
2c_1 + 2c_2 + 10c_3 + 9\sqrt{3}\,c_4 &= 3
\end{aligned}
$$

with solution $c_1 = 21/13$, $c_2 = -38/13$, $c_3 = -8/13$, $c_4 = 17\sqrt{3}/39$.



The solution is a rapidly-decaying oscillation.

38.

$$
\begin{aligned}
W(e^t, e^{-t}, \cos t, \sin t) &= -8 \\
W(\cosh t, \sinh t, \cos t, \sin t) &= 4
\end{aligned}
$$

40. Suppose that $c_1 e^{r_1 t} + c_2 e^{r_2 t} + \cdots + c_n e^{r_n t} = 0$, and each of the r_k are real and different. Multiplying this equation by $e^{-r_1 t}$, $c_1 + c_2 e^{(r_2 - r_1)t} + \cdots + c_n e^{(r_n - r_1)t} = 0$. Differentiation results in

$$c_2(r_2 - r_1)e^{(r_2 - r_1)t} + \cdots + c_n(r_n - r_1)e^{(r_n - r_1)t} = 0\,.$$

Now multiplying the latter equation by $e^{-(r_2 - r_1)t}$, and differentiating, we obtain

$$c_3(r_3 - r_2)(r_3 - r_1)e^{(r_3 - r_2)t} + \cdots + c_n(r_n - r_2)(r_n - r_1)e^{(r_n - r_2)t} = 0\,.$$

Following the above steps in a similar manner, it follows that

$$c_n(r_n - r_{n-1})\cdots(r_n - r_1)e^{(r_n - r_{n-1})t} = 0\,.$$

Since these equations hold for all t, and all the r_k are different, we have $c_n = 0$. Hence

$$c_1 e^{r_1 t} + c_2 e^{r_2 t} + \cdots + c_{n-1} e^{r_{n-1} t} = 0\,, \quad -\infty < t < \infty\,.$$

The same procedure can now be repeated, successively, to show that

$$c_1 = c_2 = \cdots = c_n = 0\,.$$

Section 4.3

2. The general solution of the homogeneous equation is $y_c = c_1e^t + c_2e^{-t} + c_3cos\,t +$ $+ c_4sin\,t$. Let $g_1(t) = 3t$ and $g_2(t) = cos\,t$. By inspection, we find that $Y_1(t) = -3t$. Since $g_2(t)$ is a solution of the homogeneous equation, set $Y_2(t) = t(Acos\,t + Bsin\,t)$. Substitution into the given ODE and comparing the coefficients of similar term results in $A = 0$ and $B = -1/4$. Hence the general solution of the nonhomogeneous problem is

$$y(t) = y_c(t) - 3t - \frac{t}{4}sin\,t.$$

3. The characteristic equation corresponding to the homogeneous problem can be written as $(r+1)(r^2+1) = 0$. The solution of the homogeneous equation is $y_c = c_1e^{-t} +$ $+ c_2cos\,t + c_3sin\,t$. Let $g_1(t) = e^{-t}$ and $g_2(t) = 4t$. Since $g_1(t)$ is a solution of the homogeneous equation, set $Y_1(t) = Ate^{-t}$. Substitution into the ODE results in $A = 1/2$. Now let $Y_2(t) = Bt + C$. We find that $B = -C = 4$. Hence the general solution of the nonhomogeneous problem is $y(t) = y_c(t) + te^{-t}/2 + 4(t-1)$.

4. The characteristic equation corresponding to the homogeneous problem can be written as $r(r+1)(r-1) = 0$. The solution of the homogeneous equation is $y_c = c_1 + c_2e^t +$ $+ c_3e^{-t}$. Since $g(t) = 2\,sin\,t$ is not a solution of the homogeneous problem, we can set $Y(t) = A\,cos\,t + B\,sin\,t$. Substitution into the ODE results in $A = 1$ and $B = 0$. Thus
the general solution is $y(t) = c_1 + c_2e^t + c_3e^{-t} + cos\,t$.

6. The characteristic equation corresponding to the homogeneous problem can be written as $(r^2+1)^2 = 0$. It follows that $y_c = c_1cos\,t + c_2sin\,t + t(c_3cos\,t + c_4sin\,t)$. Since $g(t)$ is not a solution of the homogeneous problem, set $Y(t) = A + Bcos\,2t + Csin\,2t$. Substitution into the ODE results in $A = 3$, $B = 1/9$, $C = 0$. Thus the general solution is $y(t) = y_c(t) + 3 + \frac{1}{9}cos\,2t$.

7. The characteristic equation corresponding to the homogeneous problem can be written as $r^3(r^3+1) = 0$. Thus the homogeneous solution is

$$y_c = c_1 + c_2\,t + c_3t^2 + c_4e^{-t} + e^{t/2}\left[c_5cos\left(\sqrt{3}\,t/2\right) + c_5sin\left(\sqrt{3}\,t/2\right)\right].$$

Note the $g(t) = t$ is a solution of the homogenous problem. Consider a particular solution
of the form $Y(t) = t^3(At + B)$. Substitution into the ODE results in $A = 1/24$ and $B = 0$. Thus the general solution is $y(t) = y_c(t) + t^4/24$.

8. The characteristic equation corresponding to the homogeneous problem can be written as $r^3(r+1) = 0$. Hence the homogeneous solution is $y_c = c_1 + c_2\,t + c_3t^2 + c_4e^{-t}$. Since $g(t)$ is *not* a solution of the homogeneous problem, set $Y(t) = Acos\,2t + Bsin\,2t$. Substitution into the ODE results in $A = 1/40$ and $B = 1/20$. Thus the general solution

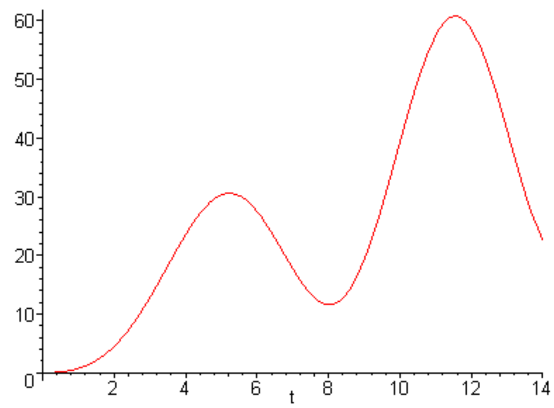
is $y(t) = y_c(t) + (cos\,2t + 2sin\,2t)/40$.

10. From Prob. 22 in Section 4.2, the homogeneous solution is

$$y_c = c_1 cos\,t + c_2 sin\,t + + t[c_3 cos\,t + c_4 sin\,t].$$

Since $g(t)$ is *not* a solution of the homogeneous problem, substitute $Y(t) = At + B$ into the ODE to obtain $A = 3$ and $B = 4$. Thus the general solution is $y(t) = y_c(t) + 3t + 4$. Invoking the initial conditions, we find that $c_1 = -4$, $c_2 = -4$, $c_3 = 1$, $c_4 = -3/2$. Therefore the solution of the initial value problem is

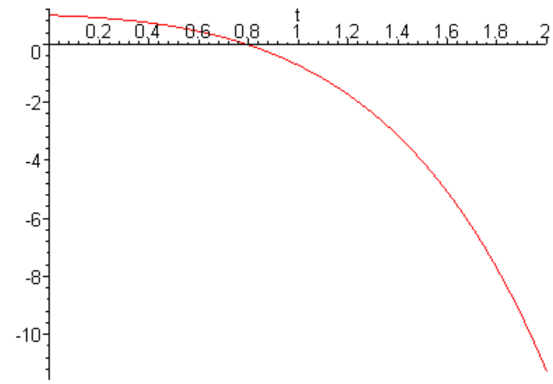
$$y(t) = (t-4)cos\,t - (3t/2 + 4)sin\,t + 3t + 4.$$



11. The characteristic equation can be written as $r(r^2 - 3r + 2) = 0$. Hence the homogeneous solution is $y_c = c_1 + c_2 e^t + c_3 e^{2t}$. Let $g_1(t) = e^t$ and $g_2(t) = t$. Note that g_1 is a solution of the homogeneous problem. Set $Y_1(t) = Ate^t$. Substitution into the ODE results in $A = -1$. Now let $Y_2(t) = Bt^2 + Ct$. Substitution into the ODE results in $B = 1/4$ and $C = 3/4$. Therefore the general solution is

$$y(t) = c_1 + c_2 e^t + c_3 e^{2t} - te^t + \left(t^2 + 3t\right)/4.$$

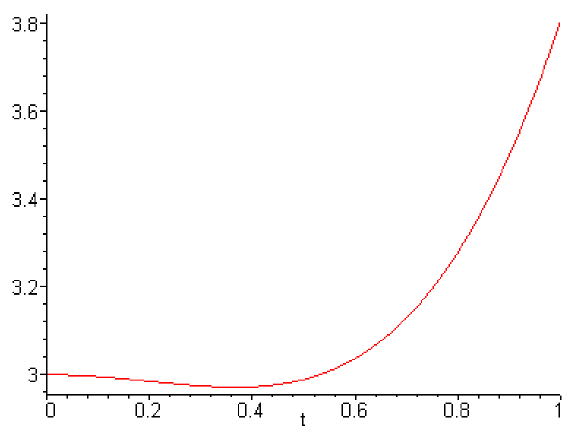
Invoking the initial conditions, we find that $c_1 = 1$, $c_2 = c_3 = 0$. The solution of the initial value problem is $y(t) = 1 - te^t + (t^2 + 3t)/4$.



12. The characteristic equation can be written as $(r-1)(r+3)(r^2+4)=0$. Hence the homogeneous solution is $y_c = c_1e^t + c_2e^{-3t} + c_3cos\,2t + c_4sin\,2t$. None of the terms in $g(t)$ is a solution of the homogeneous problem. Therefore we can assume a form $Y(t) = Ae^{-t} + Bcos\,t + Csin\,t$. Substitution into the ODE results in $A = 1/20$, $B = -2/5$, $C = -4/5$. Hence the general solution is

$$y(t) = c_1e^t + c_2e^{-3t} + c_3cos\,2t + c_4sin\,2t + e^{-t}/20 - (2cos\,t + 4sin\,t)/5\,.$$

Invoking the initial conditions, we find that $c_1 = 81/40$, $c_2 = 73/520$, $c_3 = 77/65$, $c_4 = -49/130$.



14. From Prob. 4, the homogeneous solution is $y_c = c_1 + c_2e^t + c_3e^{-t}$. Consider the terms $g_1(t) = te^{-t}$ and $g_2(t) = 2cos\,t$. Note that since $r = -1$ is a *simple* root of the characteristic equation, Table 4.3.1 suggests that we set $Y_1(t) = t(At+B)e^{-t}$. The function $2cos\,t$ is *not* a solution of the homogeneous equation. We can simply choose $Y_2(t) = Ccos\,t + Dsin\,t$. Hence the particular solution has the form

$$Y(t) = t(At+B)e^{-t} + Ccos\,t + Dsin\,t\,.$$

15. The characteristic equation can be written as $(r^2-1)^2 = 0$. The roots are given

as $r = \pm 1$, each with *multiplicity two*. Hence the solution of the homogeneous problem is $y_c = c_1 e^t + c_2 te^t + c_3 e^{-t} + c_4 te^{-t}$. Let $g_1(t) = e^t$ and $g_2(t) = sin\, t$. The function e^t is a solution of the homogeneous problem. Since $r = 1$ has multiplicity two, we set $Y_1(t) = At^2 e^t$. The function $sin\, t$ is *not* a solution of the homogeneous equation. We can set $Y_2(t) = Bcos\, t + Csin\, t$. Hence the particular solution has the form

$$Y(t) = At^2 e^t + Bcos\, t + Csin\, t\, .$$

16. The characteristic equation can be written as $r^2(r^2 + 4) = 0$, with roots $r = 0, \pm 2i$. The root $r = 0$ has multiplicity two, hence the homogeneous solution is $y_c = c_1 + c_2 t + + c_3 cos\, 2t + c_4 sin\, 2t$. The functions $g_1(t) = sin\, 2t$ and $g_2(t) = 4$ are solutions of the homogenous equation. The complex roots have multiplicity *one*, therefore we need to set $Y_1(t) = At\, cos\, 2t + Bt\, sin\, 2t$. Now $g_2(t) = 4$ is associated with the *double* root $r = 0$. Based on Table 4.3.1, set $Y_2(t) = Ct^2$. Finally, $g_3(t) = te^t$ (and its derivatives) is independent of the homogeneous solution. Therefore set $Y_3(t) = (Dt + E)e^t$. Conclude that the particular solution has the form

$$Y(t) = At\, cos\, 2t + Bt\, sin\, 2t + Ct^2 + (Dt + E)e^t.$$

18. The characteristic equation can be written as $r^2(r^2 + 2r + 2) = 0$, with roots $r = 0$, with multiplicity *two*, and $r = -1 \pm i$. The homogeneous solution is $y_c = c_1 + c_2 t + + c_3 e^{-t} cos\, t + c_4 e^{-t} sin\, t$. The function $g_1(t) = 3e^t + 2te^{-t}$, and all of its derivatives, is independent of the homogeneous solution. Therefore set $Y_1(t) = Ae^t + (Bt + C)e^{-t}$. Now $g_2(t) = e^{-t} sin\, t$ is a solution of the homogeneous equation, associated with the complex roots. We need to set $Y_2(t) = t(D\, e^{-t} cos\, t + E\, e^{-t} sin\, t)$. It follows that the particular solution has the form

$$Y(t) = Ae^t + (Bt + C)e^{-t} + t\big(D\, e^{-t} cos\, t + E\, e^{-t} sin\, t\big).$$

19. Differentiating $y = u(t)v(t)$, successively, we have

$$\begin{aligned} y' &= u'v + uv' \\ y'' &= u''v + 2u'v' + uv'' \\ &\vdots \\ y^{(n)} &= \sum_{j=0}^{n} \binom{n}{j} u^{(n-j)} v^{(j)} \end{aligned}$$

Setting $v(t) = e^{\alpha t}$, $v^{(j)} = \alpha^j e^{\alpha t}$. So for any $p = 1, 2, \cdots, n$,

$$y^{(p)} = e^{\alpha t} \sum_{j=0}^{p} \binom{p}{j} \alpha^j\, u^{(p-j)}.$$

It follows that

$$L\left[e^{\alpha t}u\right] = e^{\alpha t}\sum_{p=0}^{n}\left[a_{n-p}\sum_{j=0}^{p}\binom{p}{j}\alpha^j u^{(p-j)}\right] \qquad (*).$$

It is evident that the right hand side of Eq. $(*)$ is of the form

$$e^{\alpha t}\left[k_0\, u^{(n)} + k_1\, u^{(n-1)} + \cdots + k_{n-1}u' + k_n\, u\right].$$

Hence operator equation $L[e^{\alpha t}u] = e^{\alpha t}(b_0\, t^m + b_1\, t^{m-1} + \cdots + b_{m-1}t + b_m)$ can be written as

$$k_0\, u^{(n)} + k_1\, u^{(n-1)} + \cdots + k_{n-1}u' + k_n\, u =$$

$$= b_0\, t^m + b_1\, t^{m-1} + \cdots + b_{m-1}t + b_m\,.$$

The coefficients k_i, $i = 0, 1, \cdots, n$ can be determined by collecting the like terms in the double summation in Eq. $(*)$. For example, k_0 is the coefficient of $u^{(n)}$. The *only* term that contains $u^{(n)}$ is when $p = n$ and $j = 0$. Hence $k_0 = a_0$. On the other hand, k_n is the coefficient of $u(t)$. The inner summation in $(*)$ contains terms with u, given by $\alpha^p u$ (when $j = p$), for each $p = 0, 1, \cdots, n$. Hence

$$k_n = \sum_{p=0}^{n} a_{n-p}\, \alpha^p\,.$$

21(a). Clearly, e^{2t} is a solution of $y' - 2y = 0$, and te^{-t} is a solution of the differential equation $y'' + 2y' + y = 0$. The latter ODE has characteristic equation $(r+1)^2 = 0$. Hence $(D-2)[3e^{2t}] = 3(D-2)[e^{2t}] = 0$ and $(D+1)^2[te^{-t}] = 0$. Furthermore, we have $(D-2)(D+1)^2[te^{-t}] = (D-2)[0] = 0$, and $(D-2)(D+1)^2[3e^{2t}] =$ $= (D+1)^2(D-2)[3e^{2t}] = (D+1)^2[0] = 0$.

(b). Based on Part (a),

$$\begin{aligned}(D-2)(D+1)^2\left[(D-2)^3(D+1)Y\right] &= (D-2)(D+1)^2\left[3e^{2t} - te^{-t}\right] \\ &= 0\,,\end{aligned}$$

since the operators are linear. The implied operations are associative and commutative. Hence

$$(D-2)^4(D+1)^3 Y = 0\,.$$

The operator equation corresponds to the solution of a linear homogeneous ODE with characteristic equation $(r-2)^4(r+1)^3 = 0$. The roots are $r = 2$, with multiplicity 4 and $r = -1$, with multiplicity 3. It follows that the given homogeneous solution is

$$Y(t) = c_1 e^{2t} + c_2 te^{2t} + c_3 t^2 e^{2t} + c_4 t^3 e^{2t} + c_5 e^{-t} + c_6 te^{-t} + c_7 t^2 e^{-t},$$

which is a linear combination of seven independent solutions.

22(15). Observe that $(D-1)[e^t]=0$ and $(D^2+1)[sin\,t]=0$. Hence the operator $H(D)=(D-1)(D^2+1)$ is an annihilator of $e^t+sin\,t$. The operator corresponding to the left hand side of the given ODE is $(D^2-1)^2$. It follows that

$$(D+1)^2(D-1)^3\left(D^2+1\right)Y=0\,.$$

The resulting ODE is homogeneous, with solution

$$Y(t)=c_1e^{-t}+c_2te^{-t}+c_3e^t+c_4te^t+c_5t^3e^t+c_6cos\,t+c_7sin\,t\,.$$

After examining the homogeneous solution of Prob. 15 , and eliminating duplicate terms, we have

$$Y(t)=c_5t^3e^t+c_6cos\,t+c_7sin\,t\,.$$

22(16). We find that $D[4]=0$, $(D-1)^2[te^t]=0$, and $(D^2+4)[sin\,2t]=0$.
The operator $H(D)=D(D-1)^2(D^2+4)$is an annihilator of $t^2+te^t+sin\,2t$. The operator corresponding to the left hand side of the ODE is $D^2(D^2+4)$. It follows that

$$D^3(D-1)^2\left(D^2+4\right)^2Y=0\,.$$

The resulting ODE is homogeneous, with solution

$$Y(t)=c_1+c_2t+c_3t^2+c_4e^t+c_5te^t+c_6cos\,2t+c_7sin\,2t+c_8tcos\,2t+c_9tsin\,2t\,.$$

After examining the homogeneous solution of Prob. 16 , and eliminating duplicate terms, we have

$$Y(t)=c_3t^2+c_4e^t+c_5te^t+c_8tcos\,2t+c_9tsin\,2t\,.$$

22(18). Observe that $(D-1)[e^t]=0$, $(D+1)^2[te^{-t}]=0$. The function $e^{-t}sin\,t$ is a solution of a second order ODE with characteristic roots $r=\,-1\pm i$. It follows that $(D^2+2D+2)[e^{-t}sin\,t]=0$. Therefore the operator

$$H(D)=(D-1)(D+1)^2\left(D^2+2D+2\right)$$

is an annihilator of $3e^t+2te^{-t}+e^{-t}sin\,t$. The operator corresponding to the left hand side of the given ODE is $D^2(D^2+2D+2)$. It follows that

$$D^2(D-1)(D+1)^2\left(D^2+2D+2\right)^2Y=0\,.$$

The resulting ODE is homogeneous, with solution

$$\begin{aligned}Y(t)=\;&c_1+c_2t+c_3e^t+c_4e^{-t}+c_5te^{-t}+\\&+e^{-t}(c_6cos\,t+c_7sin\,t)+te^{-t}(c_8cos\,t+c_9sin\,t\,)\,.\end{aligned}$$

After examining the homogeneous solution of Prob. 18 , and eliminating duplicate terms, we have

$$Y(t) = c_3e^t + c_4e^{-t} + c_5te^{-t} + te^{-t}(c_8 cos\, t + c_9 sin\, t\,).$$

Section 4.4

2. The characteristic equation is $r(r^2-1)=0$. Hence the homogeneous solution is $y_c(t)=c_1+c_2e^t+c_3e^{-t}$. The Wronskian is evaluated as $W(1,e^t,e^{-t})=2$. Now compute the three determinants

$$W_1(t)=\begin{vmatrix}0 & e^t & e^{-t}\\ 0 & e^t & -e^{-t}\\ 1 & e^t & e^{-t}\end{vmatrix}=-2$$

$$W_2(t)=\begin{vmatrix}1 & 0 & e^{-t}\\ 0 & 0 & -e^{-t}\\ 0 & 1 & e^{-t}\end{vmatrix}=e^{-t}$$

$$W_3(t)=\begin{vmatrix}1 & e^t & 0\\ 0 & e^t & 0\\ 0 & e^t & 1\end{vmatrix}=e^t$$

The solution of the system of equations (10) is

$$u_1'(t)=\frac{t\,W_1(t)}{W(t)}=-t$$

$$u_2'(t)=\frac{t\,W_2(t)}{W(t)}=te^{-t}/2$$

$$u_3'(t)=\frac{t\,W_3(t)}{W(t)}=te^t/2$$

Hence $u_1(t)=-t^2/2$, $u_2(t)=-e^{-t}(t+1)/2$, $u_3(t)=e^t(t-1)/2$. The particular solution becomes $Y(t)=-t^2/2-(t+1)/2+(t-1)/2=-t^2/2-1$. The constant is a solution of the homogeneous equation, therefore the general solution is

$$y(t)=c_1+c_2e^t+c_3e^{-t}-t^2/2\,.$$

3. From Prob. 13 in Section 4.2, $y_c(t)=c_1e^{-t}+c_2e^t+c_3e^{2t}$. The Wronskian is evaluated as $W(e^{-t},e^t,e^{2t})=6\,e^{2t}$. Now compute the three determinants

$$W_1(t)=\begin{vmatrix}0 & e^t & e^{2t}\\ 0 & e^t & 2e^{2t}\\ 1 & e^t & 4e^{2t}\end{vmatrix}=e^{3t}$$

$$W_2(t) = \begin{vmatrix} e^{-t} & 0 & e^{2t} \\ -e^{-t} & 0 & 2e^{2t} \\ e^{-t} & 1 & 4e^{2t} \end{vmatrix} = -3e^{t}$$

$$W_3(t) = \begin{vmatrix} e^{-t} & e^{t} & 0 \\ -e^{-t} & e^{t} & 0 \\ e^{-t} & e^{t} & 1 \end{vmatrix} = 2$$

Hence $u_1'(t) = e^{5t}/6$, $u_2'(t) = -e^{3t}/2$, $u_3'(t) = e^{2t}/3$. Therefore the particular solution can be expressed as

$$\begin{aligned} Y(t) &= e^{-t}\left[e^{5t}/30\right] - e^{t}\left[e^{3t}/6\right] + e^{2t}\left[e^{2t}/6\right] \\ &= e^{4t}/30\,. \end{aligned}$$

6. From Prob. 22 in Section 4.2, $y_c(t) = c_1 cos\,t + c_2 sin\,t + t[c_3 cos\,t + c_4 sin\,t]$. The Wronskian is evaluated as $W(cos\,t, sin\,t, t\,cos\,t, t\,sin\,t) = 4$. Now compute the four auxiliary determinants

$$W_1(t) = \begin{vmatrix} 0 & sin\,t & t\,cos\,t & t\,sin\,t \\ 0 & cos\,t & cos\,t - t\,sin\,t & sin\,t + t\,cos\,t \\ 0 & -sin\,t & -2sin\,t - t\,cos\,t & 2cos\,t - t\,sin\,t \\ 1 & -cos\,t & -3cos\,t + t\,sin\,t & -3sin\,t - t\,cos\,t \end{vmatrix} = -2sin\,t + 2t\,cos\,t$$

$$W_2(t) = \begin{vmatrix} cos\,t & 0 & t\,cos\,t & t\,sin\,t \\ -sin\,t & 0 & cos\,t - t\,sin\,t & sin\,t + t\,cos\,t \\ -cos\,t & 0 & -2sin\,t - t\,cos\,t & 2cos\,t - t\,sin\,t \\ sin\,t & 1 & -3cos\,t + t\,sin\,t & -3sin\,t - t\,cos\,t \end{vmatrix} = 2t\,sin\,t + 2cos\,t$$

$$W_3(t) = \begin{vmatrix} cos\,t & sin\,t & 0 & t\,sin\,t \\ -sin\,t & cos\,t & 0 & sin\,t + t\,cos\,t \\ -cos\,t & -sin\,t & 0 & 2cos\,t - t\,sin\,t \\ sin\,t & -cos\,t & 1 & -3sin\,t - t\,cos\,t \end{vmatrix} = -2cos\,t$$

$$W_4(t) = \begin{vmatrix} cos\,t & sin\,t & t\,cos\,t & 0 \\ -sin\,t & cos\,t & cos\,t - t\,sin\,t & 0 \\ -cos\,t & -sin\,t & -2sin\,t - t\,cos\,t & 0 \\ sin\,t & -cos\,t & -3cos\,t + t\,sin\,t & 1 \end{vmatrix} = -2sin\,t$$

It follows that $u_1'(t) = [-sin^2 t + t\,sin\,t\,cos\,t]/2$, $u_2'(t) = [t sin^2 t + sin\,t\,cos\,t]/2$, $u_3'(t) = -sin\,t\,cos\,t/2$, $u_4'(t) = -sin^2 t/2$. Hence

$$u_1(t) = \left[3sin\,t\,cos\,t - 2t\,cos^2 t - t\right]/8$$

$$u_2(t) = \left[sin^2 t - 2cos^2 t - 2t\, sin\, t\, cos\, t + t^2\right]/8$$

$$u_3(t) = -sin^2 t/4$$

$$u_4(t) = [cos\, t\, sin\, t - t]/4$$

Therefore the particular solution can be expressed as

$$\begin{aligned} Y(t) &= cos\, t\,[u_1(t)] + sin\, t\,[u_2(t)] + t\, cos\, t\,[u_3(t)] + t\, sin\, t\,[u_4(t)] \\ &= \left[sin\, t - 3t\, cos\, t - t^2 sin\, t\right]/8\,. \end{aligned}$$

Note that only the *last term* is not a solution of the homogeneous equation. Hence the general solution is

$$y(t) = c_1 cos\, t + c_2 sin\, t + t[c_3 cos\, t + c_4 sin\, t] - t^2 sin\, t/8\,.$$

8. Based on the results in Prob. 2, $y_c(t) = c_1 + c_2 e^t + c_3 e^{-t}$. It was also shown that $W(1, e^t, e^{-t}) = 2$, with $W_1(t) = -2$, $W_2(t) = e^{-t}$, $W_3(t) = e^t$. Therefore we have $u_1'(t) = -csc\, t$, $u_2'(t) = e^{-t} csc\, t/2$, $u_3'(t) = e^t csc\, t/2$. The particular solution can be expressed as $Y(t) = [u_1(t)] + e^{-t}[u_2(t)] + e^t\,[u_3(t)]$. More specifically,

$$\begin{aligned} Y(t) &= ln|csc(t) + cot(t)| + \frac{e^t}{2}\int_{t_0}^{t} e^{-s} csc(s)ds + \frac{e^{-t}}{2}\int_{t_0}^{t} e^{s} csc(s)ds \\ &= ln|csc(t) + cot(t)| + \int_{t_0}^{t} cosh(t-s)csc(s)ds\,. \end{aligned}$$

9. Based on Prob. 4, $u_1'(t) = sec\, t$, $u_2'(t) = -1$, $u_3'(t) = -tan\, t$. The particular solution can be expressed as $Y(t) = [u_1(t)] + cos\, t\,[u_2(t)] + sin\, t\,[u_3(t)]$. That is,

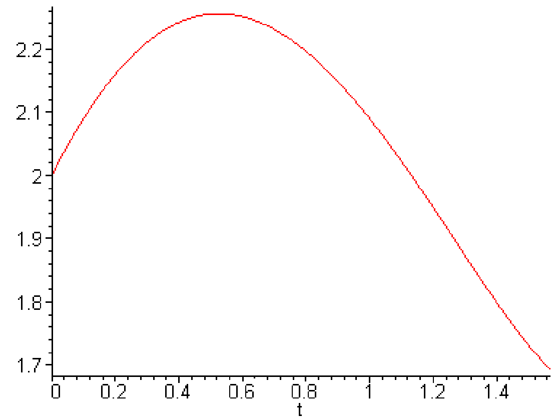
$$Y(t) = ln|sec(t) + tan(t)| - t\, cos\, t + sin\, t\, ln|cos(t)|.$$

Hence the general solution of the initial value problem is

$$y(t) = c_1 + c_2 cos\, t + c_3 sin\, t + ln|sec(t) + tan(t)| - t\, cos\, t + sin\, t\, ln|cos(t)|.$$

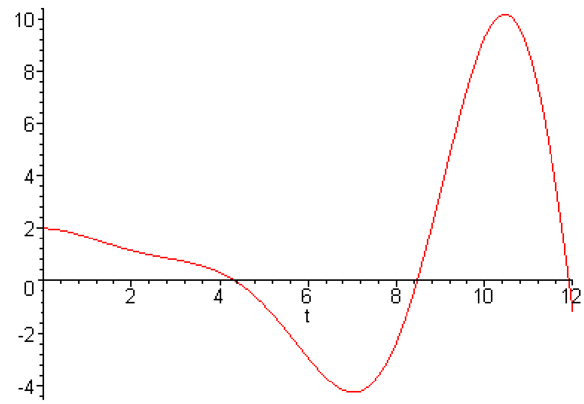
Invoking the initial conditions, we require that $c_1 + c_2 = 2$, $c_3 = 1$, $-c_2 = -2$. Therefore

$$y(t) = 2cos\, t + sin\, t + ln|sec(t) + tan(t)| - t\, cos\, t + sin\, t\, ln|cos(t)|$$



10. From Prob. 6 , $y(t) = c_1 cos\, t + c_2 sin\, t + c_3 t\, cos\, t + c_4 t\, sin\, t - t^2 sin\, t/8$. In order to satisfy the initial conditions, we require that $c_1 = 2$, $c_2 + c_3 = 0$, $-c_1 + 2c_4 = -1$, $-3/4 - c_2 - 3c_3 = 1$. Therefore

$$y(t) = 2cos\, t + \left[7sin\, t - 7t\, cos\, t + 4t\, sin\, t - t^2 sin\, t\right]/8 .$$



12. From Prob. 8 , the general solution of the initial value problem is

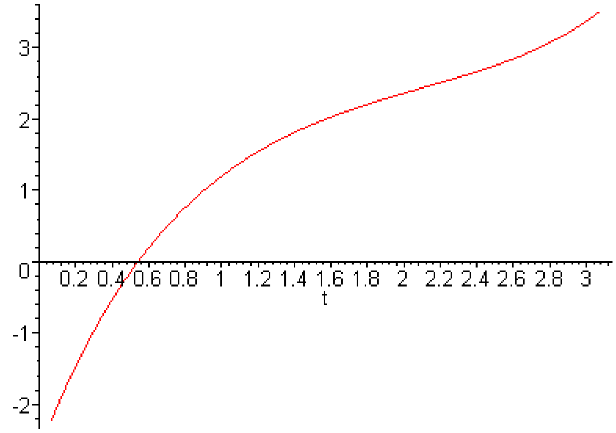
$$y(t) = c_1 + c_2 e^t + c_3 e^{-t} + ln|csc(t) + cot(t)| + \frac{e^t}{2}\int_{t_0}^t e^{-s} csc(s)ds + \frac{e^{-t}}{2}\int_{t_0}^t e^s csc(s)ds.$$

In this case, $t_0 = \pi/2$. Observe that $y(\pi/2) = y_c(\pi/2)$, $y'(\pi/2) = y_c'(\pi/2)$, and $y''(\pi/2) = y_c''(\pi/2)$. Therefore we obtain the system of equations

$$c_1 + c_2 e^{\pi/2} + c_3 e^{-\pi/2} = 2$$
$$c_2 e^{\pi/2} - c_3 e^{-\pi/2} = 1$$
$$c_2 e^{\pi/2} + c_3 e^{-\pi/2} = -1$$

Hence the solution of the initial value problem is

$$y(t) = 3 - e^{-t+\pi/2} + ln|csc(t) + cot(t)| + \int_{t_0}^{t} cosh(t-s)csc(s)ds\,.$$



13. First write the equation as $y''' + x^{-1}y'' - 2x^{-2}y' + 2x^{-3}y = 2x$. The Wronskian is evaluated as $W(x, x^2, 1/x) = 6/x$. Now compute the three determinants

$$W_1(x) = \begin{vmatrix} 0 & x^2 & 1/x \\ 0 & 2x & -1/x^2 \\ 1 & 2 & 2/x^3 \end{vmatrix} = -3$$

$$W_2(x) = \begin{vmatrix} x & 0 & 1/x \\ 1 & 0 & -1/x^2 \\ 0 & 1 & 2/x^3 \end{vmatrix} = 2/x$$

$$W_3(x) = \begin{vmatrix} x & x^2 & 0 \\ 1 & 2x & 0 \\ 0 & 2 & 1 \end{vmatrix} = x^2$$

Hence $u_1'(x) = -x^2$, $u_2'(x) = 2x/3$, $u_3'(x) = x^4/3$. Therefore the particular solution can be expressed as

$$\begin{aligned} Y(x) &= x[-x^3/3] + x^2[x^2/3] + \frac{1}{x}[x^5/15] \\ &= x^4/15\,. \end{aligned}$$

15. The homogeneous solution is $y_c(t) = c_1 cos\,t + c_2 sin\,t + c_3 cosh\,t + c_4 sinh\,t$. The Wronskian is evaluated as $W(cos\,t, sin\,t, cosh\,t, sinh\,t) = 4$. Now the four additional determinants are given by $W_1(t) = 2\,sin\,t$, $W_2(t) = -2\,cos\,t$, $W_3(t) = -2\,sinh\,t$, $W_4(t) = 2\,cosh\,t$. If follows that $u_1'(t) = g(t)\,sin(t)/2$, $u_2'(t) = -g(t)\,cos(t)/2$, $u_3'(t) = -g(t)\,sinh(t)/2$, $u_4'(t) = g(t)\,cosh(t)/2$. Therefore the particular solution

can be expressed as

$$\begin{aligned} Y(t) \;&= \frac{cos(t)}{2}\int_{t_0}^{t} g(s)\, sin(s)\, ds - \frac{sin(t)}{2}\int_{t_0}^{t} g(s)\, cos(s)\, ds - \\ &- \frac{cosh(t)}{2}\int_{t_0}^{t} g(s)\, sinh(s)\, ds + \frac{sinh(t)}{2}\int_{t_0}^{t} g(s)\, cosh(s)\, ds\,. \end{aligned}$$

Using the appropriate identities, the integrals can be combined to obtain

$$Y(t) = \frac{1}{2}\int_{t_0}^{t} g(s)\, sinh(t-s)\, ds - \frac{1}{2}\int_{t_0}^{t} g(s)\, sin(t-s)\, ds\,.$$

17. First write the equation as $y''' - 3\,x^{-1}y'' + 6\,x^{-2}y' - 6\,x^{-3}y = g(x)\,/\,x^3$. It can be shown that $y_c(x) = c_1 x + c_2\,x^2 \; + c_3\,x^3$ is a solution of the homogeneous equation. The Wronskian of this fundamental set of solutions is $W(x, x^2, x^3) = 2x^3$. The three additional determinants are given by $W_1(x) = x^4$, $W_2(x) = \,-\,2x^3$, $W_3(x) = x^2$. Hence $u_1'(x) = g(x)/2x^2$, $u_2'(x) = \,-\,g(x)\,/x^3$, $u_3'(x) = g(x)/2x^4$. Therefore the particular solution can be expressed as

$$\begin{aligned} Y(x) \;&= x\int_{x_0}^{x} \frac{g(t)}{2t^2}dt - x^2\int_{x_0}^{x} \frac{g(t)}{t^3}dt + x^3\int_{x_0}^{x} \frac{g(t)}{2t^4}dt \\ &= \frac{1}{2}\int_{x_0}^{x} \left[\frac{x}{t^2} - \frac{2\,x^2}{t^3} + \frac{x^3}{t^4}\right] g(t)dt\,. \end{aligned}$$