*C*APÍTULO 10

Exercícios 10.1

1. *a*) $f(x, y) = 5x^4y^2 + xy^3 + 4$.

Devemos olhar y como constante e derivar em relação a x:

$$\frac{\partial f}{\partial x}(x, y) = 20x^3y^2 + y^3.$$

Devemos olhar x como constante e derivar em relação a y:

$$\frac{\partial f}{\partial y}(x, y) = 10x^4y + 3xy^2.$$

***c*)**

Nos pontos $(x, y) \neq (0, 0)$ aplicamos a regra do quociente:

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x^3 + y^2}{x^2 + y^2}\right) = \frac{(x^2 + y^2)(3x^2) - (x^3 + y^2)(2x)}{(x^2 + y^2)^2} = \frac{x^4 + 3x^2y^2 - 2xy^2}{(x^2 + y^2)^2} \text{ e}$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left(\frac{x^3 + y^2}{x^2 + y^2}\right) = \frac{(x^2 + y^2)(2y) - (x^3 + y^2)\,2y}{(x^2 + y^2)^2} = \frac{2x^2y(1 - x)}{(x^2 + y^2)^2}.$$

No ponto $(x, y) = (0, 0)$ (supondo $z(0, 0) = 0$).

$\frac{\partial z}{\partial x}(0, 0)$ é a derivada, em $x = 0$, de $g(x) = z(x, 0) = x$, $x \neq 0$.

Assim $g(x) = z(x, 0) = \begin{cases} x & \text{se } x \neq 0 \\ 0 & \text{se } x = 0 \end{cases}$. Segue que $\frac{\partial z}{\partial x}(0, 0) = g'(0) = 1$.

$\frac{\partial z}{\partial y}(0, 0)$ é (caso exista) a derivada, em $y = 0$, de $h(y) = z(0, y) = 1$, $y \neq 0$.
Assim $h(y) =$

$\begin{cases} 1 & \text{se } y \neq 0 \\ 0 & \text{se } y = 0 \end{cases}$. Então $h(y)$ não é contínua em 0 e $h'(0) = \frac{\partial z}{\partial y}(0, 0)$ não existe.

d) $f(x, y) = e^{-x^2-y^2}$.

$$\frac{\partial f}{\partial x}(x, y) = \frac{\partial}{\partial x}(e^{-x^2-y^2}) = e^{-x^2-y^2} \cdot (-2x) = -2x\, e^{-x^2-y^2} \text{ e}$$
$$\frac{\partial f}{\partial y}(x, y) = \frac{\partial}{\partial y}(e^{-x^2-y^2}) = e^{-x^2-y^2} \cdot (-2y) = -2y\, e^{-x^2-y^2}$$

l) $f(x, y) = \sqrt[3]{x^3 + y^2 + 3} = (x^3 + y^2 + 3)^{\frac{1}{3}}$.

$$\frac{\partial f}{\partial x}(x, y) = \frac{\partial}{\partial x}((x^3 + y^2 + 3)^{\frac{1}{3}}) = \frac{1}{3}(x^3 + y^2 + 3)^{-\frac{2}{3}}(3x^2) = \frac{x^2}{\sqrt[3]{(x^3 + y^2 + 3)^2}} \text{ e}$$

$$\frac{\partial f}{\partial y}(x, y) = \frac{\partial}{\partial y}((x^3 + y^2 + 3)^{\frac{1}{3}}) = \frac{1}{3}(x^3 + y^2 + 3)^{-\frac{2}{3}}(2y) = \frac{2y}{3\sqrt[3]{(x^3 + y^2 + 3)^2}}.$$

m) $z = \dfrac{x \text{ sen } y}{\cos(x^2 + y^2)}$.

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x \text{ sen } y}{\cos(x^2 + y^2)}\right) = \frac{\cos(x^2 + y^2) \text{ sen } y - (x \text{ sen } y)[-\text{sen}(x^2 + y^2)(2x)]}{[\cos(x^2 + y^2)]^2}$$
$$\frac{\partial z}{\partial x} = \frac{\text{sen } y\,[\cos(x^2 + y^2) + 2x^2 \text{ sen}(x^2 + y^2)]}{[\cos(x^2 + y^2)]^2}.$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left(\frac{x \text{ sen } y}{\cos(x^2 + y^2)}\right) = \frac{\cos(x^2 + y^2)\, x \cos y - x \text{ sen } y\,[-\text{sen}(x^2 + y^2)(2y)]}{[\cos(x^2 + y^2)]^2}$$
$$\frac{\partial z}{\partial y} = \frac{x \cos y \cos(x^2 + y^2) + 2xy \text{ sen } y \text{ sen}(x^2 + y^2)}{[\cos(x^2 + y^2)]^2}.$$

3. Seja $g(x, y) = \phi\left(\dfrac{x}{y}\right)$, onde $\phi:\mathbb{R} \to \mathbb{R}$ é diferenciável e $\phi'(1) = 4$.

a) $\dfrac{\partial g}{\partial x}(x, y) = \dfrac{\partial}{\partial x}\phi\left(\dfrac{x}{y}\right) = \dfrac{1}{y}\phi'\left(\dfrac{x}{y}\right)$

$\dfrac{\partial g}{\partial x}(1, 1) = \phi'(1) = 4.$

b) $\dfrac{\partial g}{\partial y}(x, y) = \dfrac{\partial}{\partial y}\phi\left(\dfrac{x}{y}\right) = -\dfrac{x}{y^2}\phi'\left(\dfrac{x}{y}\right)$

$\dfrac{\partial g}{\partial y}(1, 1) = -\phi'(1) = -4.$

6. $pV = nRT \quad \Rightarrow \quad p = nR\dfrac{T}{V}.$

$\dfrac{\partial p}{\partial V} = nRT\,\dfrac{\partial}{\partial V}\left(\dfrac{1}{V}\right) = nRT\left(-\dfrac{1}{V^2}\right)$ (olhando n, R e T como constante e derivando em relação a V)

$$\frac{\partial p}{\partial V} = -\frac{nRT}{V^2}.$$

$\dfrac{\partial p}{\partial T} = \dfrac{nR}{V}\,\dfrac{\partial}{\partial T}(T) = \dfrac{nR}{V}$ (olhando n, R e V como constantes e derivando em relação a T).

7. Seja $z = e^y\,\phi(x-y)$

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(e^y\,\phi(x-y)) = e^y\,\phi'(x-y). \quad ①$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(e^y\,\phi(x-y)) = e^y\,\phi'(x-y)\cdot\underbrace{(-1)}_{\frac{\partial}{\partial y}(x-y)} + \phi(x-y)\,e^y. \quad ②$$

Somando ① e ②:

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = e^y\,\phi'(x-y)) - e^y\,\phi'(x-y) + \underbrace{e^y\,\phi(x-y)}_{z}$$

Logo,

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = z\cdot$$

10. Seja a equação $xyz + z^3 = x$. Derivando em relação a x (mantendo y constante):

$$xy\,\frac{\partial z}{\partial x} + yz + 3z^2\,\frac{\partial z}{\partial x} = 1$$

$$\frac{\partial z}{\partial x}(xy + 3z^2) = 1 - yz$$

$$\frac{\partial z}{\partial x} = \frac{1 - yz}{xy + 3z^2}.$$

Derivando em relação a y (olhando x como constante)

$$xy\frac{\partial z}{\partial y} + xz + 3z^2\,\frac{\partial z}{\partial y} = 0$$

$$\frac{\partial z}{\partial y}(xy + 3z^2) = -xz$$

$$\frac{\partial z}{\partial y} = -\frac{xz}{xy + 3z^2}.$$

13. Sejam $w = xy + z^4$, $z(1, 1) = 1$ e $\left.\dfrac{\partial z}{\partial x}\right|_{\substack{x=1\\y=1}} = 4$,

$$\frac{\partial w}{\partial x} = y + 4z^3\,\frac{\partial z}{\partial x}, \text{ daí}$$

$$\left.\frac{\partial w}{\partial x}\right|_{\substack{x=1\\y=1}} = 1 + 4\cdot 4 = 17.$$

15. Seja $f(x, y) = \displaystyle\int_0^{x^2+y^2} e^{-t^2}\,dt$. Considerando $F(t) = e^{-t^2}$ e $u(x, y) = x^2 + y^2$, temos

$$\frac{\partial}{\partial x}\,f(x, y) = \frac{\partial}{\partial x}\left(\int_0^{u(x,\,y)} F(t)\,dt\right) = F(u)\frac{\partial u}{\partial x} \text{ e, portanto,}$$

$$\frac{\partial}{\partial x}(x, y) = e^{-(x^2+y^2)^2}\cdot 2x = 2x\,e^{-(x^2+y^2)^2}.$$

Analogamente:

$$\frac{\partial f}{\partial y}(x, y) = F(u)\,\frac{\partial u}{\partial y}, \text{ ou seja,}$$

$$\frac{\partial f}{\partial y}(x, y) = 2y\,e^{-(x^2+y^2)^2}.$$

16. $f(x, y) = \displaystyle\int_{x^2}^{y^2} e^{-t^2}\,dt = \int_{x^2}^{0} e^{-t^2}\,dt + \int_0^{y^2} e^{-t^2}\,dt$

$$= -\int_0^{x^2} e^{-t^2}\,dt + \int_0^{y^2} e^{-t^2}\,dt.$$

Considerando $F(t) = e^{-t^2}$, $u(x) = x^2$ e $v(y) = y^2$, temos:

$$\frac{\partial f}{\partial x}(x, y) = \frac{\partial}{\partial x}\left(-\int_0^{x^2} e^{-t^2}\,dt\right) + \frac{\partial}{\partial x}\left(\int_0^{y^2} e^{-t^2}\,dt\right)$$

$$= -F(u)\frac{du}{dx} + F(v)\,\frac{dv}{dx}. \text{ Portanto,}$$

$$\frac{\partial f}{\partial x}(x, y) = -2x\,e^{-x^4}.$$

$$\frac{\partial f}{\partial y}(x, y) = \frac{\partial}{\partial y}\left[-\int_0^{x^2} e^{-t^2}\,dt\right] + \frac{\partial}{\partial y}\left[\int_0^{y^2} e^{-t^2}\,dt\right] =$$

$= -F(u)\,\dfrac{du}{dy} + F(v)\,\dfrac{dv}{dy}$. Logo,

$$\frac{\partial f}{\partial y}(x, y) = 2y\, e^{-y^4}$$

18. Seja $f(x, y) = x^3y^2 - 6\,xy + \phi(y)$. Temos

$$\frac{\partial f}{\partial y} = 2x^3y - 6x + \phi'(y)$$

Comparando com $\dfrac{\partial f}{\partial y} = 2x^3y - 6x + \dfrac{y}{y^2+1}$, resulta

$\phi'(y) = \dfrac{y}{y^2+1}$. Daí

$$\phi(y) = \int \frac{y\,dy}{y^2+1} = \frac{1}{2}\ln\,(1 + y^2) + c.$$

Portanto,

$$\phi(y) = \frac{1}{2}\ln\,(1 + y^2).$$

21. *b*) $\dfrac{\partial f}{\partial x}(x, y) = \dfrac{\partial}{\partial x}\left(e^{\frac{1}{x^2+y^2-1}}\right) = e^{\left(\frac{1}{x^2+y^2-1}\right)} \cdot \dfrac{\partial}{\partial x}\left(\dfrac{1}{x^2+y^2-1}\right)$

$\dfrac{\partial f}{\partial x}(x, y) = -\dfrac{2x}{(x^2+y^2-1)^2}\, e^{\left(\frac{1}{x^2+y^2-1}\right)}$, se $x^2 + y^2 < 1$ e

$\dfrac{\partial f}{\partial x}(x, y) = 0$ se $x^2 + y^2 > 1$. Para $x^2 + y^2 = 1$, $\dfrac{\partial f}{\partial x}(x,\ y)$ tem que ser calculado pela definição. Lembrando que $f(x, y) = 0$ para $x^2 + y^2 = 1$, temos

$$\frac{\partial f}{\partial x}(x,\ y) = \lim_{u \to x} \frac{f(u,\ y) - f(x,\ y)}{u - x}$$

$= \lim\limits_{u \to x} \dfrac{f(u,\ y)}{u - x}$. Para $|\,u\,| > |\,x\,|$, $f(u,\ y) = 0$, logo tal limite é zero. Para $|\,u\,| < |\,x\,|$,

$$\frac{\partial f}{\partial x}(x,\ y) = \lim_{u \to x} \frac{e^{\frac{1}{u^2 + y^2 - 1}}}{u - x} = \left[\frac{0}{0}\right].$$

Aplicando L'Hospital,

$$\frac{\partial f}{\partial x}(x,\ y) = \lim_{u \to x} e^{\frac{1}{u^2 + y^2 - 1}}\ \frac{-2u}{(u^2 + y^2 - 1)^2}.$$

Fazendo $s = \dfrac{1}{u^2 + y^2 - 1}$, para $u \to x$, $s \to -\infty$, temos

$$\lim_{u \to x} e^{\frac{1}{u^2 + y^2 - 1}} \frac{1}{(u^2 + y^2 - 1)^2} = \lim_{s \to -\infty} e^s s^2$$

$$= \lim_{s \to -\infty} \frac{s^2}{e^{-s}} = 0.$$

Daí, para $| u | < | x |$

$$\frac{\partial f}{\partial x}(x, y) = -\lim_{u \to x} e^{\frac{1}{u^2 + y^2 - 1}} \frac{1}{(u^2 + y^2 - 1)^2} \lim_{u \to x} 2u = 0.$$

Assim, $\dfrac{\partial f}{\partial x}(x, y) = 0$ para $x^2 + y^2 = 1$.

Conclusão:

$$\frac{\partial f}{\partial x}(x, y) = \begin{cases} \dfrac{-2x}{(x^2 + y^2 - 1)} e^{\frac{1}{x^2 + y^2 - 1}} & \text{se } x^2 + y^2 < 1 \\ 0 & \text{se } x^2 + y^2 \geqslant 1 \end{cases}$$

Do mesmo modo mostra-se que

$$\frac{\partial f}{\partial x}(x, y) = \begin{cases} \dfrac{-2y}{(x^2 + y^2 - 1)^2} e^{\frac{1}{x^2 + y^2 - 1}} & \text{se } x^2 + y^2 < 1 \\ 0 & \text{se } x^2 + y^2 \geqslant 1 \end{cases}$$

23.

a) $z(t) = f(t, t) = t^2 + t^2 = 2t^2$

c) $\gamma(t) = (t, t, 2t^2)$.

$\gamma'(t) = (1, 1, 4t) \Rightarrow \gamma'(1) = (1, 1, 4)$

Reta tangente a γ no ponto $(1, 1, 2)$

$(x, y, z) = (1, 1, 2) + \lambda(1, 1, 4)$.

d) Seja o plano $z - f(1, 1) = \dfrac{\partial f}{\partial x}(1, 1)(x - 1) + \dfrac{\partial f}{\partial y}(1, 1)(y - 1)$.

O ponto $(1, 1, 2)$ pertence ao plano.

O vetor $\left(\dfrac{\partial f}{\partial x}(1, 1), \dfrac{\partial f}{\partial y}(1, 1), -1\right) = (2, 2, -1)$ é normal ao plano. Agora $(1, 1, 4) \cdot (2, 2, -1) = 0$. Portanto, o vetor $\gamma'(1) = (1, 1, 4)$ é ortogonal ao vetor $(2, 2, -1)$

normal ao plano. Logo, a reta tangente T: $(x, y, z) = (1, 1, 2) + \lambda(1, 1, 4)$ está contida no plano de equação

$$z - f(1,1) = \frac{\partial f}{\partial x}(1, 1)(x - 1) + \frac{\partial f}{\partial y}(1, 1)(y - 1).$$

29. ***a)*** $f(x, y) = x^2 + y^2$. Temos

$$\begin{cases} \dfrac{\partial f}{\partial x}(x, y) = 2x \\ \dfrac{\partial f}{\partial y}(x, y) = 2y \end{cases}$$

Resolvendo o sistema:

$$\begin{cases} 2x = 0 \\ 2y = 0 \end{cases} \quad \Rightarrow \quad (x, y) = (0, 0) \text{ é ponto crítico ou estacionário}.$$

f) $f(x, y) = x^4 + 4xy + y^4$. Temos

$$\begin{cases} \dfrac{\partial f}{\partial x} = 4x^3 + 4y \\ \dfrac{\partial f}{\partial y} = 4y^3 + 4x. \end{cases} \qquad \begin{cases} x^3 + y = 0 \\ y^3 + x = 0 \end{cases} \quad \Rightarrow \quad y = -x^3.$$

Portanto,

$$\begin{cases} x = 0 \\ x = \pm 1. \end{cases}$$

Daí $\begin{cases} y = 0 \\ y = \mp 1. \end{cases}$

Pontos críticos (0, 0) ; (1, −1) ; (−1, 1).

Exercícios 10.2

1. ***a)*** Seja $f(x, y, z) = x\,e^{x-y-z}$. Temos

$\dfrac{\partial f}{\partial x}(x, y, z) = xe^{x-y-z} + e^{x-y-z} = (x + 1)\,e^{x-y-z}$ (y e z são olhadas como constantes),

$\dfrac{\partial f}{\partial y}(x, y, z) = -xe^{x-y-z}$ (x e z são olhadas como constantes) e

$\dfrac{\partial f}{\partial z}(x, y, z) = -x\,e^{x-y-z}$ (x e y são olhadas como constantes).

c) $w = \dfrac{xyz}{x + y + z}$.

$$\frac{\partial w}{\partial x} = \frac{(x + y + z)\, yz - xyz}{(x + y + z)^2} = \frac{y + z}{(x + y + z)^2},$$

$$\frac{\partial w}{\partial y} = \frac{(x + y + z)\, xz - xyz}{(x + y + z)^2} = \frac{x + z}{(x + y + z)^2} \text{ e}$$

$$\frac{\partial w}{\partial z} = \frac{(x + y + z)\, xy - xyz}{(x + y + z)^2} = \frac{x + y}{(x + y + z)^2}.$$

e)

$s = xw \ln (x^2 + y^2 + z^2 + w^2)$

$\frac{\partial s}{\partial x} = w\left[\ln(x^2 + y^2 + z^2 + w^2) + \frac{2x^2}{x^2+y^2+z^2+w^2}\right]$ (y, z, w são olhadas como constantes),

$$\frac{\partial s}{\partial y} = \frac{2xyw}{x^2+y^2+z^2+w^2},$$

$$\frac{\partial s}{\partial z} = \frac{2xzw}{x^2 + y^2 + z^2 + w^2} \text{ e}$$

$$\frac{\partial s}{\partial w} = x\left[\ln(x^2 + y^2 + z^2 + w^2) + \frac{2w^2}{x^2+y^2+z^2+w^2}\right].$$

4. Sejam $g(x, y, z) = \int_0^{x+y^2+z^4} f(t)\, dt$ e $f : \mathbb{R} \to \mathbb{R}$ contínua com $f(3) = 4$.

c) $\frac{\partial g}{\partial z}(x, y, z) = f(u)\,\frac{du}{dz}$ onde $u = x + y^2 + z^4$. Assim,

$\frac{\partial g}{\partial z}(x, y, z) = f(x + y^2 + z^4) \cdot 4z^3$. Daí

$\frac{\partial g}{\partial z}(1, 1, 1) = f(3) \cdot 4 = 4 \cdot 4 = 16$.

6. Sejam ϕ: $\mathbb{R} \to \mathbb{R}$ diferenciável tal que $\phi'(3) = 4$ e $g(x, y, z) = \phi(x^2 + y^2 + z^2)$

a) $\frac{\partial g}{\partial x}(x, y, z) = \phi'(x^2 + y^2 + z^2) \cdot 2x$

$\frac{\partial g}{\partial x}(1, 1, 1) = \phi'(3) \cdot 2 = 4 \cdot 2 = 8$.

b) $\frac{\partial g}{\partial y}(x, y, z) = \phi'(x^2+y^2+z^2) \cdot 2y$

$\frac{\partial g}{\partial y}(1, 1, 1) = \phi'(3) \cdot 2 = 8$.

c) $\frac{\partial g}{\partial z}(x, y, z) = \phi'(x^2 + y^2 + z^2) \cdot 2z$

$\frac{\partial g}{\partial z}(1, 1, 1) = \phi'(3) \cdot 2 = 8$.