CAPÍTULO 14

Exercícios 14.1

1. ***b***) $\dfrac{\partial z}{\partial x} = 2x\, e^{x^2-y^2}$ e $\dfrac{\partial z}{\partial y} = -2y\, e^{x^2-y^2}$. Temos

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}(2xe^{x^2-y^2}) = 4x^2e^{x^2-y^2} + 2e^{x^2-y^2} = 2e^{x^2-y^2}(2x^2+1),$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}(-2ye^{x^2-y^2}) = 4y^2e^{x^2-y^2} - 2e^{x^2-y^2} = 2e^{x^2-y^2}(2y^2-1),$$

$$\frac{\partial^2 z}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}(-2ye^{x^2-y^2}) = -4xye^{x^2-y^2} \text{ e}$$

$$\frac{\partial^2 z}{\partial y\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}(2xe^{x^2-y^2}) = -4xye^{x^2-y^2}.$$

c) $\dfrac{\partial z}{\partial x} = \dfrac{2x}{1+x^2+y^2}$ e $\dfrac{\partial z}{\partial y} = \dfrac{2y}{1+x^2+y^2}$. Temos

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{2x}{1+x^2+y^2}\right) = \frac{(1+x^2+y^2)\cdot 2 - 2x\cdot 2x}{(1+x^2+y^2)^2},$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{2+2y^2-2x^2}{(1+x^2+y^2)^2},$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{2y}{1+x^2+y^2}\right) = \frac{2+2x^2-2y^2}{(1+x^2+y^2)^2} \text{ e}$$

$$\frac{\partial^2 z}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{2y}{1+x^2+y^2}\right) = -\frac{4xy}{(1+x^2+y^2)^2} = \frac{\partial^2 z}{\partial y\partial x}.$$

2. Seja $f(x, y) = \dfrac{1}{x^2+y^2}$.

a) $\dfrac{\partial f}{\partial x} = -\dfrac{2x}{(x^2+y^2)^2}$; $\dfrac{\partial^2 f}{\partial x^2} = \dfrac{(x^2+y^2)^2\cdot(-2) + 2x\cdot 2(x^2+y^2)\cdot 2x}{(x^2+y^2)^4}$, ou seja,

$$\frac{\partial^2 f}{\partial x^2} = \frac{6x^4+4x^2y^2-2y^4}{(x^2+y^2)^4}.$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left(-\frac{2x}{(x^2+y^2)^2}\right) = \frac{2x \cdot 2(x^2+y^2) \cdot 2y}{(x^2+y^2)^4}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{8xy(x^2+y^2)}{(x^2+y^2)^4}.$$

Substituindo,

$$x\frac{\partial^2 f}{\partial x^2}(x,\ y) + y\frac{\partial^2 f}{\partial y \partial x}(x,\ y)$$

$$= \frac{6x^5 + 4x^3y^2 - 2xy^4 + 8x^3y^2 + 8xy^4}{(x^2+y^2)^4} = \frac{6x^5 + 12x^3y^2 + 6xy^4}{(x^2+y^2)^4}$$

$$= \frac{6x(x^4 + 2x^2y^2 + y^4)}{(x^2+y^2)^4} = \frac{6x(x^2+y^2)^2}{(x^2+y^2)^4} = \frac{6x}{(x^2+y^2)^2}$$

$$= (-3) \cdot -\frac{2x}{(x^2+y^2)^2} = -3\frac{\partial f}{\partial x}(x,\ y).$$

Logo, $x\dfrac{\partial^2 f}{\partial x^2}(x, y) + \dfrac{y\partial^2 f}{\partial y \partial x}(x, y) = -3\dfrac{\partial f}{\partial x}(x, y)$. (**Observação.** Poderíamos ter chegado a este resultado sem fazer contas: é só observar que $\dfrac{\partial f}{\partial x}$ é homogênea de grau -3 e usar a relação de Euler.)

***b*)** $\dfrac{\partial^2 f}{\partial x^2} = \dfrac{6x^4 + 4x^2y^2 - 2y^4}{(x^2+y^2)^4}$ e

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(-\frac{2y}{(x^2+y^2)^2}\right) = \frac{(x^2+y^2)^2 \cdot (-2) + 2y\,2(x^2+y^2) \cdot 2y}{(x^2+y^2)^4}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{-2x^4 - 4x^2y^2 - 2y^4 + 8x^2y^2 + 8y^4}{(x^2+y^2)^4} = \frac{-2x^4 + 4x^2y^2 + 6y^4}{(x^2+y^2)^4}.$$

Substituindo,

$$\frac{\partial^2 f}{\partial y^2}(x,\ y) + \frac{\partial^2 f}{\partial y^2}(x,\ y) = \frac{6x^4 + 4x^2y^2 - 2y^4 - 2x^4 + 4x^2y^2 + 6y^4}{(x^2+y^2)^4}$$

$$= \frac{4x^4 + 8x^2y^2 + 4y^4}{(x^2+y^2)^4} = \frac{4(x^2+y^2)^2}{(x^2+y^2)^4} = \frac{4}{(x^2+y^2)^2}.$$

Portanto, a identidade se verifica:

$$\frac{\partial^2 f}{\partial x^2}(x,\ y) + \frac{\partial^2 f}{\partial y^2}(x,\ y) = \frac{4}{(x^2+y^2)^2}.$$

3. $f(x,\ y) = \ln(x^2 + y^2)$.

$$\frac{\partial f}{\partial x} = \frac{2x}{x^2 + y^2}; \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{2x}{x^2 + y^2}\right) = \frac{(x^2 + y^2) \cdot 2 - 2x \cdot 2x}{(x^2 + y^2)^2},$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{2y^2 - 2x^2}{(x^2 + y^2)^2},$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{2y}{x^2 + y^2}\right) = \frac{2x^2 - 2y^2}{(x^2 + y^2)^2} \text{ e daí}$$

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{2y^2 - 2x^2 + 2x^2 - 2y^2}{(x^2 + y^2)^2} = 0.$$

5. Como f, g: $A \subset \mathbb{R}^2 \to \mathbb{R}$, A aberto, são funções de classe C^2**,**

conclui-se que $\dfrac{\partial^2 g}{\partial x \partial y} = \dfrac{\partial^2 g}{\partial y \partial x}$ e $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x}$. Temos

$$\frac{\partial f}{\partial x} = \frac{\partial g}{\partial y} \quad \Rightarrow \quad \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial g}{\partial y}\right) = \frac{\partial^2 g}{\partial x \partial y} \text{ e}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(-\frac{\partial g}{\partial x}\right) = -\frac{\partial^2 g}{\partial y \partial x} = -\frac{\partial^2 g}{\partial x \partial y}.$$

Portanto,

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 g}{\partial x \partial y} - \frac{\partial^2 g}{\partial x \partial y} = 0.$$

Analogamente,

$$\frac{\partial^2 g}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial g}{\partial x}\right) = \frac{\partial}{\partial x}\left(-\frac{\partial f}{\partial y}\right) = -\frac{\partial^2 f}{\partial x \partial y} \text{ e}$$

$$\frac{\partial^2 g}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial g}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}.$$

Portanto,

$$\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} = -\frac{\partial^2 f}{\partial x \partial y} + \frac{\partial^2 f}{\partial x \partial y} = 0.$$

6. Como f: $A \subset \mathbb{R}^3 \to \mathbb{R}$ é de classe C^2 no aberto A (f e todas as derivadas parciais de 1.ª e 2.ª ordens são contínuas), pelo teorema de Schwarz, temos:

a) $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x}$

b) $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial z \partial x}$

c) $\dfrac{\partial^2 f}{\partial y \partial z} = \dfrac{\partial^2 f}{\partial z \partial y}$

8. Seja $f(x, y) = \begin{cases} xy\,\dfrac{x^2 - y^2}{x^2 + y^2} & \text{se } (x, y) \neq (0, 0) \\ 0 & \text{se } (x, y) = (0, 0) \end{cases}$

Devemos, inicialmente, determinar as derivadas parciais de 1.ª ordem de *f*.
Para $(x, y) \neq (0, 0)$ temos:

$$\frac{\partial f}{\partial x} = \frac{4x^2y^3 + x^4y - y^5}{(x^2 + y^2)^2}$$

e

$$\frac{\partial f}{\partial y} = \frac{-4x^3y^2 - xy^4 + x^5}{(x^2 + y^2)^2}.$$

Em (0, 0) temos:

$$\frac{\partial f}{\partial x}(0, 0) = \lim_{x \to 0} \frac{f(x, 0) - f(0, 0)}{x} = 0 \text{ e}$$

$$\frac{\partial f}{\partial y}(0, 0) = \lim_{y \to 0} \frac{f(0, y) - f(0, 0)}{y} = 0.$$

Portanto,

$$\frac{\partial f}{\partial x}(x, y) = \begin{cases} \dfrac{4x^2y^3 + x^4y - y^5}{(x^2 + y^2)^2} & \text{se } \ (x, y) \neq (0, 0) \\ 0 & \text{se } \ (x, y) = (0, 0) \end{cases}$$

$$\frac{\partial f}{\partial y}(x, y) = \begin{cases} \dfrac{-4x^3y^2 - xy^4 + x^5}{(x^2 + y^2)^2} & \text{se } \ (x, y) \neq (0, 0) \\ 0 & \text{se } \ (x, y) = (0, 0). \end{cases}$$

Continuando, calculando agora:

$$\frac{\partial^2 f}{\partial x \partial y}(0, 0) = \lim_{x \to 0} \frac{\dfrac{\partial f}{\partial y}(x, 0) - \dfrac{\partial f}{\partial y}(0, 0)}{x - 0}$$

$$= \lim_{x \to 0} \frac{\dfrac{x^5}{x^4} - 0}{x} = \lim_{x \to 0} \frac{x}{x} = 1.$$

$$\frac{\partial^2 f}{\partial y \partial x}(0, 0) = \lim_{y \to 0} \frac{\dfrac{\partial f}{\partial x}(0, y) - \dfrac{\partial f}{\partial y}(0, 0)}{y - 0}$$

$$= \lim_{y\to 0} \frac{-\frac{y^5}{y^4} - 0}{y} = \lim_{y\to 0} -\frac{y}{y} = -1.$$

Logo, $\frac{\partial^2 f}{\partial x \partial y}(0, 0) = 1$ e $\frac{\partial^2 f}{\partial y \partial x}(0, 0) = -1$. ($f(x, y)$ é de classe C^2 em $\mathbb{R}^2$? Por quê?)

9. Seja $u(x, t) = A \text{ sen}(a\lambda t + \varphi) \text{ sen } \lambda x$.

$$\frac{\partial u}{\partial t} = A \text{ sen } \lambda x \cdot \cos(a\lambda t + \varphi) \cdot (a\lambda)$$

$$\frac{\partial^2 u}{\partial t^2} = -A\,(a\lambda) \text{ sen } \lambda x \cdot (-\text{sen}\,(a\lambda t + \varphi))\; a\lambda.$$

$$\frac{\partial^2 u}{\partial t^2} = -Aa^2\lambda^2 \text{ sen } \lambda x \cdot \text{sen}(a\lambda t + \varphi). \qquad ①$$

Por outro lado,

$$\frac{\partial u}{\partial x} = A \text{ sen }(a\lambda t + \varphi)(\cos \lambda x) \cdot \lambda$$

$$\frac{\partial^2 u}{\partial x^2} = -A\lambda^2 \text{ sen }(a\lambda t + \varphi) \cdot \text{sen } \lambda x. \quad ②$$

Comparando ① e ②:

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}.$$

10. Seja $u = f(x - at) + g(x + at)$.
Considerando $y = x - at$ e $z = x + at$.

$$\frac{\partial u}{\partial t} = \frac{df}{dy}\frac{\partial y}{\partial t} + \frac{dg}{dz}\frac{\partial z}{\partial t} \quad \text{e} \quad \frac{\partial u}{\partial x} = \frac{df}{dy}\frac{\partial y}{\partial x} + \frac{dg}{dz}\frac{\partial z}{\partial x}$$

$$\frac{\partial u}{\partial t} = -a\frac{df}{dy} + a\frac{dg}{dz} \quad \text{e} \quad \frac{\partial u}{\partial x} = \frac{df}{dy} + \frac{dg}{dz}$$

$$\frac{\partial^2 u}{\partial t^2} = a^2\frac{d^2 f}{dy^2} + a^2\frac{d^2 g}{dz^2} \quad \text{e} \quad \frac{\partial^2 u}{\partial x^2} = \frac{d^2 f}{dy^2} + \frac{d^2 g}{dz^2}.$$

Portanto,

$$\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2}.$$

11. Sejam $x = x(u, v)$ e $y = y(u, v)$ com $x(1, 1) > 0$,
$x^3 + y^3 = u - v$ e
$xy = u - 2v$.

Calculando as derivadas parciais,

$$\begin{cases} 3x^2 \dfrac{\partial x}{\partial u} + 3y^2 \dfrac{\partial y}{\partial u} = 1 \\ x \dfrac{\partial y}{\partial u} + y \dfrac{\partial x}{\partial u} = 1. \end{cases}$$

Resolvendo o sistema, temos:

$$\frac{\partial x}{\partial u} = \frac{x - 3y^2}{3x^3 - 3y^3}$$

Se $xy = u - 2v$, então $(x(1, 1)) \cdot (y(1, 1)) = 1 - 2 = -1$.

Mas $x(1, 1) > 0$. Logo, $y(1, 1) < 0$ e $y(1, 1) = -\dfrac{1}{x(1, 1)}$ ①

Se $x^3 + y^3 = u - v$, então $(x(1, 1))^3 + (y(1, 1))^3 = 0$.
Logo, $y(1, 1) = -x(1, 1)$ ②
De ① e ② concluímos que $x(1, 1) = 1$ e $y(1, 1) = -1$.

Portanto, $\dfrac{\partial x}{\partial u}(1,1) = \dfrac{x(1, 1) - 3[y(1, 1)]^2}{3[x(1, 1)]^3 - 3[y(1, 1)]^3} = \dfrac{1-3}{3+3} = -\dfrac{2}{6} = -\dfrac{1}{3}$.

14. Seja $z = \displaystyle\int_1^{x^2-y^2} \left[\int_0^u \text{sen } t^2 \, dt \right] du.$

Pelo teorema fundamental do Cálculo temos:

$$\frac{\partial z}{\partial y} = \left[\int_0^{x^2-y^2} \text{sen } t^2 \, dt \right] (-2y)$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = (-2y)\,\text{sen}(x^2 - y^2)^2 \cdot 2x$$

$$\frac{\partial^2 z}{\partial x \partial y} = -4xy\,\text{sen}(x^2 - y^2)^2.$$

b) $\dfrac{\partial^2 z}{\partial x^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial x}\right)$

$$\frac{\partial z}{\partial x} = 2x \int_0^{x^2-y^2} \text{sen } t^2 \, dt$$

$$\frac{\partial^2 z}{\partial x^2} = 4x^2 \,\text{sen}\,(x^2 - y^2)^2 + 2\int_0^{x^2-y^2} \text{sen } t^2 \, dt$$

$$\frac{\partial^2 z}{\partial x^2}(1,1) = 4 \cdot 1 \cdot \underbrace{\text{sen } 0}_{0} + 2 \underbrace{\int_0^0 \text{sen } t^2 \, dt}_{0}$$

$$\frac{\partial^2 z}{\partial x^2}(1,1) = 0.$$

Exercícios 14.2

1.

a) $g(t) = \frac{\partial f}{\partial x}(x, y)$ com $x = t^2$ e $y = \text{sen } t$. Temos

$$\frac{dx}{dt} = 2t \quad \text{e} \quad \frac{dy}{dt} = \cos t.$$

$$g'(t) = \frac{d}{dt}\left[\frac{\partial f}{\partial x}(x, y)\right] = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)\frac{dx}{dt} + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\frac{dy}{dt}, \text{ ou seja,}$$

$$g'(t) = 2t\,\frac{\partial^2 f}{\partial x^2}(x, y) + \cos t\,\frac{\partial^2 f}{\partial y \partial x}(x, y).$$

b) $g(t) = t^3 \frac{\partial f}{\partial x}(x, y)$, com $x = 3t$ e $y = 2t$. Temos

$$\frac{dx}{dt} = 3 \quad \text{e} \quad \frac{dy}{dt} = 2.$$

$$g'(t) = \frac{d}{dt}\left[t^3 \frac{\partial f}{\partial x}(3t, 2t)\right],$$

$$g'(t) = 3t^2 \frac{\partial f}{\partial x}(3t, 2t) + t^3\left[\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}(3t, 2t)\right)\frac{dx}{dt} + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}(3t, 2t)\right)\frac{dy}{dt}\right], \text{ ou seja,}$$

$$g'(t) = 3t^2 \frac{\partial f}{\partial x}(3t, 2t) + 3t^3 \frac{\partial^2 f}{\partial x^2}(3t, 2t) + 2\frac{\partial^2 f}{\partial y \partial x}(3t, 2t).$$

c) $g(t) = \frac{\partial f}{\partial x}(t^2, 2t) + 5\frac{\partial f}{\partial y}(\text{sen } 3t, t)$ daí

$$g'(t) = \frac{d}{dt}\left[\frac{\partial f}{\partial x}(t^2, 2t)\right] + 5\frac{d}{dt}\left[\frac{\partial f}{\partial y}(\text{sen } 3t, t)\right], \text{ ou seja,}$$

$$g'(t) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}(t^2, 2t)\right) \cdot 2t + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}(t^2, 2t)\right) \cdot 2 + 5\left[\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}(\text{sen } 3t, t)\right)\right] 3\cos 3t$$

$$+ \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}(\text{sen } 3t, t)\right) = 2t\,\frac{\partial^2 f}{\partial x^2}(t^2, 2t) + 2\frac{\partial^2 f}{\partial y \partial x}(t^2, 2t)$$

$$+ 15\cos 3t\,\frac{\partial^2 f}{\partial x \partial y}(\text{sen } 3t, t) + 5\frac{\partial^2 f}{\partial y^2}(\text{sen } 3t, t).$$

3. Seja $g(t) = f(a + ht, b + kt)$.

a) $f(x, y)$ é de classe C^2 (f admite as derivadas parciais de 1.ª ordem e 2.ª ordem contínuas).

$g'(t) = \dfrac{d}{dt}\left[f(x, y)\right]$, com $x = a + ht$ e $y = b + kt$. Temos

$\dfrac{dx}{dt} = h$ e $\dfrac{dy}{dt} = k$. Segue que

$$g'(t) = \frac{\partial f}{\partial x}(x, y)\,\frac{dx}{dt} + \frac{\partial f}{\partial y}(x, y)\,\frac{dy}{dt},$$ ou seja,

$$g'(t) = h\frac{\partial f}{\partial x}(x, y) + k\frac{\partial f}{\partial y}(x, y).$$ Temos

$$g''(t) = \frac{d}{dt}\left[h\frac{\partial f}{\partial x}(x, y) + k\frac{\partial f}{\partial y}(x, y)\right],$$ ou seja,

$$g''(t) = h\left[\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}(x, y)\right)\frac{dx}{dt} + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}(x, y)\right)\frac{dy}{dt}\right]$$

$$+ k\left[\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}(x, y)\right)\frac{dx}{dt} + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}(x, y)\right)\frac{dy}{dt}\right].$$ Assim,

$$g''(t) = h\left[h\frac{\partial^2 f}{\partial x^2}(x, y) + k\frac{\partial^2 f}{\partial y \partial x}(x, y)\right] + k\left[h\frac{\partial^2 f}{\partial x \partial y}(x, y) + k\frac{\partial^2 f}{\partial y^2}(x, y)\right],$$ ou seja,

$$g''(t) = h^2\frac{\partial^2 f}{\partial x^2}(x, y) + hk\frac{\partial^2 f}{\partial y \partial x}(x, y) + hk\frac{\partial^2 f}{\partial x \partial y}(x, y) + k^2\frac{\partial^2 f}{\partial y^2}(x, y).$$

Pelo teorema de Schwarz (f é de classe C^2),

$$\frac{\partial^2 f}{\partial x \partial y}(x, y) = \frac{\partial^2 f}{\partial y \partial x}(x, y).$$

Portanto,

$$g''(t) = h^2\frac{\partial^2 f}{\partial x^2}(x, y) + 2hk\frac{\partial^2 f}{\partial x \partial y}(x, y) + k^2\frac{\partial^2 f}{\partial y^2}(x, y).$$

b) Supondo $f(x, y)$ de classe C^3 num aberto de $\mathbb{R}^2$ (f admite todas as derivadas de ordem 3 contínuas no aberto de $\mathbb{R}^2$).

$$g'''(t) = \frac{d}{dt}\left[h^2\frac{\partial^2 f}{\partial x^2}(x, y) + 2hk\frac{\partial^2 f}{\partial x \partial y}(x, y) + k^2\frac{\partial^2 f}{\partial y^2}(x, y)\right],$$ ou seja,

$$g'''(t) = h^2\frac{d}{dt}\left(\frac{\partial^2 f}{\partial x^2}(x, y)\right) + 2hk\frac{d}{dt}\left(\frac{\partial^2 f}{\partial x \partial y}(x, y)\right) + k^2\frac{d}{dt}\left(\frac{\partial^2 f}{\partial y^2}(x, y)\right). \quad ①$$

Temos:

$$\frac{d}{dt}\left(\frac{\partial^2 f}{\partial x^2}(x, y)\right) = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x^2}(x, y)\right)\frac{dx}{dt} + \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial x^2}(x, y)\right)\frac{dy}{dt}, \text{ ou seja,}$$

$$\frac{d}{dt}\left(\frac{\partial^2 f}{\partial x^2}(x, y)\right) = h\frac{\partial^3 f}{\partial x^3}(x, y) + k\frac{\partial^3 f}{\partial y \partial x^2}(x, y). \quad ②$$

$$\frac{d}{dt}\left(\frac{\partial^2 f}{\partial x \partial y}(x, y)\right) = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x \partial y}(x, y)\right)\frac{dx}{dt} + \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial x \partial y}(x, y)\right)\frac{dy}{dt}, \text{ ou seja,}$$

$$\frac{d}{dt}\left(\frac{\partial^2 f}{\partial x \partial y}(x, y)\right) = h\frac{\partial^3 f}{\partial x^2 \partial y}(x, y) + k\frac{\partial^3 f}{\partial y^2 \partial x}(x, y). \quad ③$$

$$\frac{d}{dt}\left(\frac{\partial^2 f}{\partial y^2}(x, y)\right) = \frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial y^2}(x, y)\right)\frac{dx}{dt} + \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial y^2}(x, y)\right)\frac{dy}{dt}, \text{ ou seja,}$$

$$\frac{d}{dt}\left(\frac{\partial^2 f}{\partial y^2}(x, y)\right) = h\frac{\partial^3 f}{\partial x \partial y^2}(x, y) + k\frac{\partial^3 f}{\partial y^3}(x, y). \quad ④$$

Substituindo ②, ③ e ④ em ①,

$$g'''(t) = h^3\frac{\partial^3 f}{\partial x^3}(x, y) + h^2 k\frac{\partial^3 f}{\partial y \partial x^2}(x, y) + 2h^2 k\frac{\partial^3 f}{\partial x^2 \partial y}(x, y)$$
$$+ 2hk^2\frac{\partial^3 f}{\partial x \partial y^2}(x, y) + hk^2\frac{\partial^3 f}{\partial x \partial y^2}(x, y) + k^3\frac{\partial^3 f}{\partial y^3}(x, y).$$

Como f é de classe C^3, temos: $\frac{\partial^3 f}{\partial y \partial x^2} = \frac{\partial^3 f}{\partial x^2 y}$; $\frac{\partial^3 f}{\partial x \partial y^2} = \frac{\partial^3 f}{\partial y^2 \partial x}$.

Portanto,

$$g'''(t) = h^3\frac{\partial^3 f}{\partial x^3}(x, y) + 3h^2 k\frac{\partial^3 f}{\partial x^2 \partial y}(x, y) + 3hk^2\frac{\partial^3 f}{\partial x \partial y^2}(x, y) + k^3\frac{\partial^3 f}{\partial y^3}(x, y).$$

5. $z = \frac{\partial f}{\partial x}(x, y)$, onde $y = \text{sen } 3x$

$$\frac{dz}{dx} = \frac{d}{dx}\left(\frac{\partial f}{\partial x}(x, \text{sen } 3x)\right)$$

$$\frac{dz}{dx} = \frac{\partial^2 f}{\partial x^2}(x, \text{sen } 3x)\frac{dx}{dx} + \frac{\partial^2 f}{\partial y \partial x}(x, \text{sen } 3x)\frac{dy}{dx}$$

$$\frac{dz}{dx} = \frac{\partial^2 f}{\partial x^2}(x, \text{sen } 3x) + 3\cos 3x\,\frac{\partial^2 f}{\partial y \partial x}(x, \text{sen } 3x).$$

7. $g(u, v) = f(x, y)$, com $x = 2u + v$ e $y = u - 2v$.

$\frac{dg}{\partial u} = \frac{\partial}{\partial u}[f(x, y)] = \frac{\partial f}{\partial x}(x, y)\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}(x, y)\frac{\partial y}{\partial u}$, ou seja,

$\frac{dg}{\partial u} = 2\frac{\partial}{\partial u} + \frac{\partial f}{\partial y}$. Segue que

$\frac{\partial^2 g}{\partial u^2} = \frac{\partial}{\partial u}\left[2\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\right] = 2\frac{\partial}{\partial u}\left(\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial u}\left(\frac{\partial f}{\partial y}\right)$. Temos

$\frac{\partial}{\partial u}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)\frac{\partial x}{\partial u} + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\frac{\partial y}{\partial u}$, ou seja,

$\frac{\partial}{\partial u}\left(\frac{\partial f}{\partial x}\right) = 2\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y \partial x}$;

$\frac{\partial}{\partial u}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\frac{\partial x}{\partial u} + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)\frac{\partial y}{\partial u}$, ou seja,

$\frac{\partial}{\partial u}\left(\frac{\partial f}{\partial y}\right) = 2\frac{\partial^2 f}{\partial x \partial y} + \frac{\partial^2 f}{\partial y}$.

Logo, substituindo e aplicando o teorema de Schwarz,

$$\frac{\partial^2 g}{\partial u^2} = 4\frac{\partial^2 f}{\partial x^2} + 4\frac{\partial^2 f}{\partial x \partial y} + \frac{\partial^2 f}{\partial y^2}. \quad ①$$

Procedendo de forma análoga, obtém-se

$$\frac{\partial^2 g}{\partial v^2} = \frac{\partial^2 f}{\partial x^2} - 4\frac{\partial^2 f}{\partial x \partial y} + 4\frac{\partial^2 f}{\partial y^2}. \quad ②$$

Somando ① e ② segue:

$$\frac{\partial^2 g}{\partial u^2} + \frac{\partial^2 g}{\partial v^2} = 5\frac{\partial^2 f}{\partial x^2} + 5\frac{\partial^2 f}{\partial y^2}.$$

8. *Sugestão*. Calcule

$$\frac{\partial v}{\partial r}\left(\frac{\partial v}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}\right), \frac{\partial^2 v}{\partial r^2}, \frac{\partial v}{\partial \theta}\left(\frac{\partial v}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}\right) \text{ e } \frac{\partial^2 v}{\partial \theta^2},$$

em seguida, calcule a soma

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial r^2}.$$

10.

a) Seja $g(u, v) = f(x, t)$, onde $x = u + v$ e $t = u - v$.

Temos $\frac{\partial x}{\partial u} = 1$; $\frac{\partial x}{\partial v} = 1$; $\frac{\partial t}{\partial u} = 1$ e $\frac{\partial t}{\partial v} = -1$.

E mais: $\dfrac{\partial g}{\partial u} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial u} + \dfrac{\partial f}{\partial t}\dfrac{\partial t}{\partial u} = \dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial t}$,

$$\frac{\partial^2 g}{\partial v \partial u} = \frac{\partial}{\partial v}\left(\frac{\partial g}{\partial u}\right) = \frac{\partial}{\partial v}\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial t}\right) = \frac{\partial}{\partial v}\left(\frac{\partial f}{\partial x}\right)$$

$$+ \frac{\partial}{\partial v}\left(\frac{\partial f}{\partial t}\right) = \frac{\partial^2 f}{\partial x^2}\frac{\partial x}{\partial v} + \frac{\partial^2 f}{\partial t \partial x}\frac{\partial t}{\partial v} + \frac{\partial^2 f}{\partial x \partial t}\frac{\partial x}{\partial v} + \frac{\partial^2 f}{\partial t^2}\frac{\partial t}{\partial v}$$

$$= \frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial t \partial x} + \frac{\partial^2 f}{\partial x \partial t} - \frac{\partial^2 f}{\partial t^2} = \left(\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial t^2}\right) + \left(\frac{\partial^2 f}{\partial x \partial t} - \frac{\partial^2 f}{\partial t \partial x}\right).$$

Como $f(x, t)$ satisfaz à equação $\dfrac{\partial^2 f}{\partial x^2} = \dfrac{\partial^2 f}{\partial t^2} \Rightarrow \dfrac{\partial^2 f}{\partial x^2} - \dfrac{\partial^2 f}{\partial t^2} = 0.$

Além disso, f é de classe C^2, logo vale o teorema de Schwarz.

$$\frac{\partial^2 f}{\partial x \partial t} = \frac{\partial^2 f}{\partial t \partial x} \Rightarrow \frac{\partial^2 f}{\partial x \partial t} - \frac{\partial^2 f}{\partial t \partial x} = 0.$$

Portanto,

$$\frac{\partial^2 g}{\partial v \partial u} = \left(\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial t^2}\right) + \left(\frac{\partial^2 f}{\partial x \partial t} - \frac{\partial^2 f}{\partial t \partial x}\right) = 0.$$

***b*)** De $\dfrac{\partial}{\partial v}\left(\dfrac{\partial g}{\partial u}\right) = 0$ segue que $\dfrac{\partial g}{\partial u}$ não depende de v, assim, deveremos ter $g(u, v) = \theta(u) + \varphi(v)$, com $\theta(u)$ e $\varphi(v)$ deriváveis até a segunda ordem.
Assim, $f(x, t) = \theta(x + t) + \varphi(x - t)$ satisfaz ①. Por exemplo, $f(x, t) = \cos(x + t) + \text{sen}(x - t)$ é solução da equação;
$f(x, t) = (x + t)^3 - 5(x + t)^2 + e^{(x - t)^3}$ é, também, solução etc.

11.

***a*)** Seja $g(u, v) = f(x, t)$, onde $x = mu + nv$ e $t = pu + qv$.
Temos

$$\frac{\partial^2 g}{\partial u \partial v} = \frac{\partial}{\partial u}\left(\frac{\partial g}{\partial v}\right) = \frac{\partial}{\partial u}\left(\frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial t}\frac{\partial t}{\partial v}\right) = \frac{\partial}{\partial u}\left(n\frac{\partial f}{\partial x} + q\frac{\partial f}{\partial t}\right)$$

$$= n\frac{\partial^2 f}{\partial x^2}\frac{\partial x}{\partial u} + q\frac{\partial^2 f}{\partial x \partial t}\frac{\partial x}{\partial u} + n\frac{\partial^2 f}{\partial t \partial x}\frac{\partial t}{\partial u} + q\frac{\partial^2 f}{\partial t^2}\frac{\partial t}{\partial u}.$$

Portanto,

$$\frac{\partial^2 g}{\partial u \partial v} = mn\frac{\partial^2 f}{\partial x^2} + pq\frac{\partial^2 f}{\partial t^2} + np\frac{\partial^2 f}{\partial t \partial x} + mq\frac{\partial^2 f}{\partial x \partial t}.$$

De $\dfrac{\partial^2 f}{\partial t^2} = c^2\dfrac{\partial^2 f}{\partial x^2}$ $(c \neq 0)$. Segue

$$\frac{\partial^2 g}{\partial u \partial v} = (mn + pqc^2)\frac{\partial^2 f}{\partial x^2} + (np + mq)\frac{\partial^2 f}{\partial x \partial t}.$$

Para que ocorra $mn + pqc^2 = 0$ e $np + mq = 0$, basta tomar $m = c$, $n = c$, $p = 1$ e $q = 1$.

b) $f(x,\ t) = F(x - ct) + G(x + ct)$, com $F(u)$ e $G(v)$ deriváveis até a 2.ª ordem resolve o problema.

13. Sejam $z = z(x,\ y)$; $x = e^u \cos v$ e $y = e^u \operatorname{sen} v$.

Logo, $\dfrac{\partial x}{\partial u} = e^u \cos v$; $\dfrac{\partial x}{\partial v} = -\, e^u \operatorname{sen} v$; $\dfrac{\partial y}{\partial u} = e^u \operatorname{sen} v$; $\dfrac{\partial y}{\partial v} = e^u \cos v$

Temos, $\dfrac{\partial z}{\partial u} = e^u \cos v \dfrac{\partial z}{\partial x} + e^u \operatorname{sen} v \dfrac{\partial z}{\partial y}$ e

$$\frac{\partial^2 z}{\partial u^2} = e^u \cos v \frac{\partial z}{\partial x} + e^u \cos v \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial x}\right) + e^u \operatorname{sen} v \frac{\partial z}{\partial y} + e^u \operatorname{sen} v \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial y}\right).$$

Tendo em vista que:

$$\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)\frac{\partial x}{\partial u} + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)\frac{\partial y}{\partial u} = e^u \cos v \frac{\partial^2 z}{\partial x^2} + e^u \operatorname{sen} v \frac{\partial^2 z}{\partial y \partial x} \text{ e}$$

$$\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)\frac{\partial x}{\partial u} + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)\frac{\partial y}{\partial u} = e^u \cos v \frac{\partial^2 z}{\partial x \partial y} + e^u \operatorname{sen} v \frac{\partial^2 z}{\partial y^2}$$

resulta:

$$\frac{\partial^2 z}{\partial u^2} = e^u \cos v \frac{\partial z}{\partial x} + e^{2u} \cos^2 v \frac{\partial^2 z}{\partial x^2} + e^{2u} \operatorname{sen} v \cos v \frac{\partial^2 z}{\partial y \partial x}$$
$$+ e^u \operatorname{sen} v \frac{\partial z}{\partial y} + e^{2u} \operatorname{sen} v \cos v \frac{\partial^2 z}{\partial x \partial y} + e^{2u} \operatorname{sen}^2 v \frac{\partial^2 z}{\partial y^2}. \quad ①$$

Procedendo de forma análoga, obtemos:

$$\frac{\partial^2 z}{\partial v^2} = -e^u \cos v \frac{\partial z}{\partial x} + e^{2u} \operatorname{sen}^2 v \frac{\partial^2 z}{\partial x^2} - e^{2u} \operatorname{sen} v \cos v \frac{\partial^2 z}{\partial y \partial x}$$
$$-e^u \operatorname{sen} v \frac{\partial z}{\partial y} - e^{2u} \operatorname{sen} v \cos v \frac{\partial^2 z}{\partial x \partial y} + e^{2u} \cos^2 v \frac{\partial^2 z}{\partial y^2}. \quad ②$$

Somando ① e ②:

$$\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} = e^{2u} (\cos^2 v + \operatorname{sen}^2 v) \frac{\partial^2 z}{\partial x^2} + e^{2u} (\operatorname{sen}^2 v + \cos^2 v) \frac{\partial^2 z}{\partial y^2}$$

$$\Rightarrow \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} = e^{2u} \underbrace{\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right)}_{0} \Rightarrow \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} = 0.$$

14. Seja $G(u,\ v) = \dfrac{F(x,\ y)}{x}$, com $u = x + y$ e $v = \dfrac{y}{x}$.

Derivando $u = x + y$ e $v = \dfrac{y}{x}$ em relação a v (u constante).

$$\begin{cases} 0 = \dfrac{\partial x}{\partial v} + \dfrac{\partial y}{\partial v} \\ 1 = -\dfrac{y}{x}\dfrac{\partial x}{\partial v} + \dfrac{1}{x}\dfrac{\partial y}{\partial v} \end{cases}$$

Resolvendo o sistema $\dfrac{\partial x}{\partial v} = -\dfrac{x^2}{x+y}$ e $\dfrac{\partial y}{\partial v} = \dfrac{x^2}{x+y}$.

Derivando $G(u, v) = \dfrac{F(x, y)}{x}$ em relação a v:

$$\frac{\partial G}{\partial v} = \frac{\partial}{\partial v}\left(\frac{F}{x}\right) = \frac{\partial}{\partial x}\left(\frac{F}{x}\right)\frac{\partial x}{\partial v} + \frac{\partial}{\partial y}\left(\frac{F}{x}\right)\frac{\partial y}{\partial v}$$

$$\frac{\partial G}{\partial v} = \left(\frac{1}{x}\frac{\partial F}{\partial x} - \frac{F}{x^2}\right)\left(-\frac{x^2}{x+y}\right) + \left(\frac{x}{x^2}\frac{\partial F}{\partial y}\right)\left(\frac{x^2}{x+y}\right)$$

$$\frac{\partial G}{\partial v} = \left(\frac{x}{x+y}\right)\left(\frac{\partial F}{\partial y} - \frac{\partial F}{\partial x}\right) + \frac{F}{x+y}.$$

Derivando novamente em relação a v:

$$\frac{\partial^2 G}{\partial v^2} = \frac{\partial}{\partial v}\left[\frac{x}{x+y}\left(\frac{\partial F}{\partial y} - \frac{\partial F}{\partial x}\right) + \frac{F}{x+y}\right]$$

$$\frac{\partial^2 G}{\partial v^2} = \frac{\partial}{\partial x}\left[\frac{x}{x+y}\left(\frac{\partial F}{\partial y} - \frac{\partial F}{\partial x}\right) + \frac{F}{x+y}\right]\frac{\partial x}{\partial v} + \frac{\partial}{\partial y}\left[\frac{x}{x+y}\left(\frac{\partial F}{\partial y} - \frac{\partial F}{\partial x}\right) + \frac{F}{x+y}\right]\frac{\partial y}{\partial v}$$

$$\begin{aligned}\frac{\partial^2 G}{\partial v^2} = &\left[\frac{x}{x+y}\left(\frac{\partial^2 F}{\partial x\partial y^2} - \frac{\partial^2 F}{\partial x^2}\right) + \left(\frac{\partial F}{\partial y} - \frac{\partial F}{\partial x}\right)\cdot\frac{y}{(x+y)^2} + \frac{1}{(x+y)}\frac{\partial F}{\partial x}\right.\\ &\left. - \frac{F}{(x+y)^2}\right]\left(-\frac{x^2}{x+y}\right) + \left[\frac{x}{x+y}\left(\frac{\partial^2 F}{\partial y^2} - \frac{\partial^2 F}{\partial y\partial x}\right)\right.\\ &\left. + \left(\frac{\partial F}{\partial y} - \frac{\partial F}{\partial x}\right)\left(-\frac{x}{(x+y)^2}\right) + \frac{1}{(x+y)}\frac{\partial F}{\partial y} - \frac{F}{(x+y)^2}\right]\frac{x^2}{(x+y)}\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 G}{\partial v^2} = &\frac{-x^3}{(x+y)^2}\frac{\partial^2 F}{\partial x\partial y} + \frac{x^3}{(x+y)^2}\frac{\partial^2 F}{\partial x^2} - \frac{x^2 y}{(x+y)^3}\frac{\partial F}{\partial y} + \frac{x^2 y}{(x+y)^3}\frac{\partial F}{\partial x}\\ &- \frac{x^2}{(x+y)^2}\frac{\partial F}{\partial x} + \frac{x^2 F}{(x+y)^3} + \frac{x^3}{(x+y)^2}\frac{\partial^2 F}{\partial y^2} - \frac{x^3}{(x+y)^2}\frac{\partial^2 F}{\partial y\partial x}\\ &- \frac{x^3}{(x+y)^3}\frac{\partial F}{\partial y} + \frac{x^3}{(x+y)^3}\frac{\partial F}{\partial x} + \frac{x^2}{(x+y)^2}\frac{\partial F}{\partial y} - \frac{x^2 F}{(x+y)^3}\end{aligned}$$

$$\frac{\partial^2 G}{\partial v^2} = \frac{x^3}{(x+y)^2}\underbrace{\left(\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} - \frac{2\partial^2 F}{\partial x\partial y}\right)}_{0} + \underbrace{\frac{x^2 y + x^3 - x^2(x+y)}{(x+y)^3}}_{0}\left(\frac{\partial F}{\partial x} - \frac{\partial F}{\partial y}\right)$$

$$\Rightarrow \frac{\partial^2 G}{\partial v^2} = 0.$$