CAPÍTULO 2

Exercícios 2.1

1.

a) $\int_0^2 f(x)\, dx$, onde $f(x) = \begin{cases} 2 & \text{se} \quad 0 \leqslant x < 1 \\ \dfrac{1}{x} & \text{se} \quad 1 \leqslant x \leqslant 2 \end{cases}$

f é integrável em [0, 2], pois é limitada e descontínua apenas em $x = 1$.

Temos $\int_0^2 f(x)\, dx = \int_0^1 f(x)\, dx + \int_1^2 f(x)\, dx$

Em [0, 1], $f(x)$ difere de 2 apenas em $x = 1$. Daí,

$$\int_0^1 f(x)\, dx = \int_0^1 2\, dx = [2x]_0^1 = 2$$

Em [1, 2], $f(x) = \dfrac{1}{x}$. Logo,

$$\int_0^1 f(x)\, dx = \int_1^2 \frac{dx}{x} = [\ln x]_1^2 = \ln 2$$

Portanto, $\int_0^2 f(x)\, dx = 2 + \ln 2$.

c) $\int_{-1}^3 f(x)\, dx$, onde $f(x) = \begin{cases} \dfrac{x}{1+x^2} & \text{se} \quad x \neq 1 \\ 5 & \text{se} \quad x = 1 \end{cases}$

f é integrável em $[-1, 3]$, pois é limitada e descontínua apenas em $x = 1$.

$$\begin{aligned} \int_{-1}^3 f(x)\, dx &= \int_{-1}^1 \frac{x}{1+x^2}\, dx + \int_1^3 \frac{x}{1+x^2}\, dx \\ &= \frac{1}{2}\left[\ln(1+x^2)\right]_{-1}^1 + \frac{1}{2}\left[\ln(1+x^2)\right]_1^3 \\ &= \frac{1}{2}[\ln 2 - \ln 2] + \frac{1}{2}[\ln 10 - \ln 2] \\ &= \frac{1}{2}\ln 5. \end{aligned}$$

d) $\int_{-2}^{2} g(u)\, du$, onde $g(u) = \begin{cases} \dfrac{1}{u^2} & \text{se } |u| \geqslant 1 \\ u & \text{se } |u| < 1 \end{cases}$

g é integrável em $[-2, 2]$ pois g é contínua em $[-2, 2]$.

Temos $g(u) = \begin{cases} \dfrac{1}{u^2} & \text{se } u \leqslant -1 \\ u & \text{se } -1 < u < 1 \\ \dfrac{1}{u^2} & \text{se } u \geqslant 1 \end{cases}$

Logo,

$$\begin{aligned} \int_{-2}^{2} g(u)\, du &= \int_{-2}^{-1} \frac{du}{u^2} + \int_{-1}^{1} u\, du + \int_{1}^{2} \frac{du}{u^2} \\ &= \left[-\frac{1}{u}\right]_{-2}^{-1} + \left[\frac{u^2}{2}\right]_{-1}^{1} + \left[-\frac{1}{u}\right]_{1}^{2} \\ &= 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{2} - \frac{1}{2} + 1 = 1. \end{aligned}$$

2.

b) $\int_{-1}^{x} f(t)\, dt$, onde $f(t) = \begin{cases} t^2 & \text{se } -1 \leqslant t \leqslant 1 \\ 2 & \text{se } t > 1 \end{cases}$

Para todo $x \geqslant -1$, f é integrável em $[-1, x]$, pois, neste intervalo, f é limitada e descontínua no máximo em um ponto. Temos:

$$\int_{-1}^{x} f(t)\, dt = \begin{cases} \displaystyle\int_{-1}^{x} t^2\, dt & \text{se } -1 \leqslant x \leqslant 1 \\ \displaystyle\int_{-1}^{1} t^2\, dt + \int_{1}^{x} 2\, dt & \text{se } x > 1 \end{cases}$$

Como $\int_{-1}^{x} t^2\, dt = \frac{x^3}{3} + \frac{1}{3}$; $\int_{-1}^{1} t^2\, dt = \frac{2}{3}$ e $\int_{1}^{x} 2\, dt = 2x - 2$
segue que:

$$\int_{-1}^{x} f(t)\, dt = \begin{cases} \dfrac{x^3}{3} + \dfrac{1}{3} & \text{se } -1 \leqslant x \leqslant 1 \\ 2x - \dfrac{4}{3} & \text{se } x > 1 \end{cases}$$

c) $\int_{0}^{x} f(t)\, dt$, onde $f(t) = \begin{cases} t^2 & \text{se } -1 \leqslant t \leqslant 1 \\ 2 & \text{se } t > 1 \end{cases}$

Para todo $x \geqslant -1$ f é integrável em $[x, 0]$, se $x \leqslant 0$, e em $[0, x]$, se $x > 0$, pois nestes intervalos f é limitada e descontínua no máximo em um ponto.

$$\int_0^x f(t)\,dt = \begin{cases} \int_0^x t^2\,dt & \text{se } -1 \leqslant x \leqslant 1 \\ \int_0^1 t^2\,dt + \int_1^x 2\,dt & \text{se } x > 1 \end{cases}$$

Temos $\int_0^x t^2\,dt = \frac{x^3}{3}$; $\int_0^1 t^2\,dt = \frac{1}{3}$; $\int_1^x 2\,dt = 2x - 2$.

Logo,

$$\int_0^x f(t)\,dt = \begin{cases} \frac{x^3}{3} & \text{se } -1 \leqslant x \leqslant 1 \\ 2x - \frac{5}{3} & \text{se } x > 1 \end{cases}$$

Exercícios 2.2

1.

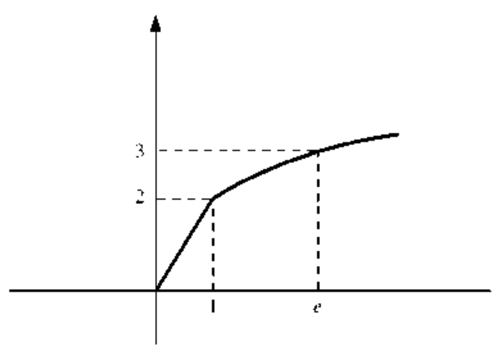
a) $F(x) = \int_0^x f(t)\,dt$, onde $f(t) = \begin{cases} 2 & \text{se } 0 \leqslant t < 1 \\ \frac{1}{t} & \text{se } t \geqslant 1 \end{cases}$

O domínio de F é intervalo $[0, +\infty[$. Temos:

$$F(x) = \int_0^x f(t)\,dt = \begin{cases} \int_0^x 2\,dt & \text{se } 0 \leqslant x < 1 \\ \int_0^1 2\,dt + \int_1^x \frac{dt}{t} & \text{se } x \geqslant 1 \end{cases}$$

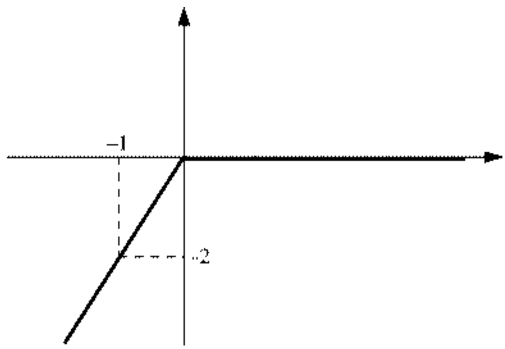
Então,

$$F(x) = \begin{cases} 2x & \text{se } 0 \leqslant x < 1 \\ 2 + \ln x & \text{se } x \geqslant 1 \end{cases}$$



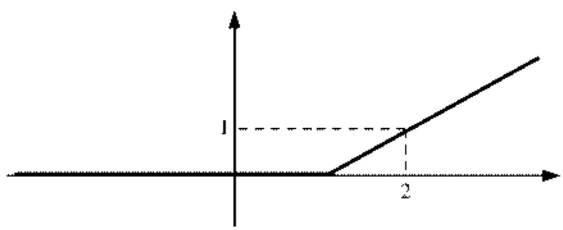
c) $F(x)=\int_1^x f(t)\,dt$, onde $f(t)=\begin{cases}2 & \text{se } t\leqslant 0\\ 0 & \text{se } t>0.\end{cases}$

$$F(x)=\begin{cases}\int_0^x 2\,dt & \text{se } x\leqslant 0\\ 0 & \text{se } x>0\end{cases} \quad \text{ou} \quad F(x)=\begin{cases}2x & \text{se } x\leqslant 0\\ 0 & \text{se } x>0\end{cases}$$



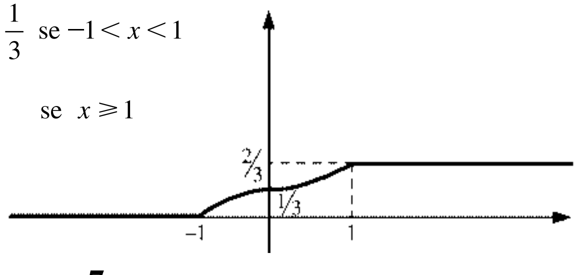
d) $F(x)=\int_1^x f(t)\,dt$ onde $f(t)=\begin{cases}0 & \text{se } t\leqslant 1\\ 1 & \text{se } t>1\end{cases}$

$$F(x)=\begin{cases}0 & \text{se } x\leqslant 1\\ \int_1^x dt=x-1 & \text{se } x>1\end{cases}$$



f) $F(x)=\int_{-5}^x f(t)\,dt$, onde $f(t)=\begin{cases}0 & \text{se } t\leqslant -1\\ t^2 & \text{se } -1<t<1\\ 0 & \text{se } t>1\end{cases}$

$$F(x)=\begin{cases}0 & \text{se } x\leqslant -1\\ \int_{-1}^x t^2\,dt=\left[\dfrac{t^3}{3}\right]_{-1}^x=\dfrac{x^3}{3}+\dfrac{1}{3} & \text{se } -1<x<1\\ \int_{-1}^1 t^2\,dt+\int_1^x 0\,dt=\dfrac{2}{3} & \text{se } x\geqslant 1\end{cases}$$

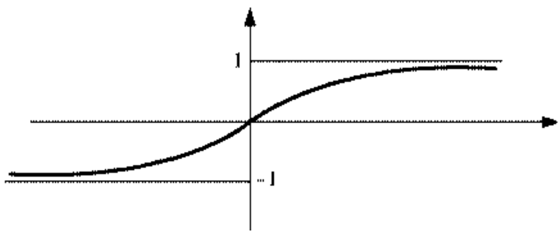


g) $F(x)=\int_0^x e^{-|t|}\,dt \qquad |t|=\begin{cases} t & \text{se } t \geqslant 0 \\ -t & \text{se } t<0. \end{cases}$

$$F(x)=\begin{cases} \int_0^x e^t\,dt & \text{se } x<0 \\ \int_0^x e^{-t}\,dt & \text{se } x \geqslant 0 \end{cases}$$

Como $\int_0^x e^t\,dt = e^x - 1$ e $\int_0^x e^{-t}\,dt = -e^{-x}+1$, temos:

$$F(x)=\begin{cases} e^x - 1 & \text{se } x<0 \\ -e^{-x}+1 & \text{se } x \geqslant 0 \end{cases}$$

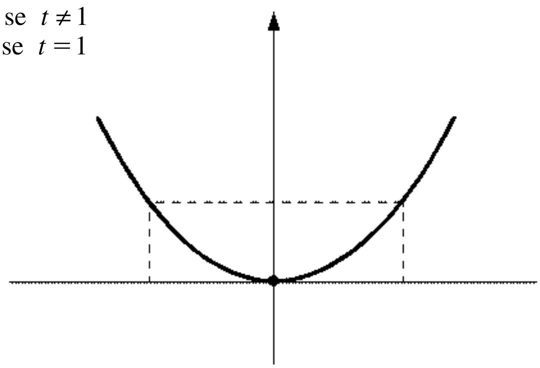


2. $F(x)=\int_0^x f(t)\,dt$, onde $f(t)=\begin{cases} t & \text{se } t \neq 1 \\ 2 & \text{se } t=1 \end{cases}$

a) $F(x)=\int_0^1 t\,dt+\int_1^x t\,dt$

$F(x)=\left[\dfrac{t^2}{2}\right]_0^1+\dfrac{x^2}{2}-\dfrac{1}{2}$

$F(x)=\dfrac{x^2}{2},\ x \in \mathbb{R}$



b) $F'(x) = x \in \mathbb{R}$. (Observe: F é derivável em $x = 1$, embora f não seja contínua neste ponto.)

3.

a) $F(x)=\int_2^x \underbrace{\frac{1}{t-1}}_{f(t)}\,dt$

$\int_2^x f(t)\,dt$ existe para todo $x > 1$.

Se $x \leqslant 1$ $\int_2^x f(t)\,dt$ não existe.
Logo, $D_F =]1, +\infty[$

d) $F(x) = \int_3^x \underbrace{\frac{1}{t^2-4}}_{f(t)}\,dt$

$\int_3^x f(t)\,dt$ existe para todo $x > 2$.
$D_F =]2, +\infty[$.

4. $F(x) = \int_0^x f(t)\,dt$, onde $f(t) = \begin{cases} t^2 & \text{se } t < 1 \\ \frac{2}{t} & \text{se } t \geqslant 1 \end{cases}$

(f não é contínua em $t = 1$)

a)

$$F(x) = \begin{cases} \int_0^x t^2\,dt & \text{se } x < 1 \\ \int_0^1 t^2\,dt + \int_1^x \frac{2}{t}\,dt & \text{se } x \geqslant 1 \end{cases}$$

$$F(x) = \begin{cases} \frac{x^3}{3} & \text{se } x < 1 \\ \frac{1}{3} + 2\ln x & \text{se } x \geqslant 1 \end{cases}$$

$$F'(x) = \begin{cases} x^2 & \text{se } x < 1 \\ \frac{2}{x} & \text{se } x > 1 \end{cases}$$

b) $F(x)$ não é derivável em $x = 1$, pois

$$\lim_{x \to 1^-} \frac{F(x) - F(1)}{x - 1} = 1 \text{ e } \lim_{x \to 1^+} \frac{F(x) - F(1)}{x - 1} = 2.$$

Exercícios 2.4

1.

a) $F(x) = \int_{-2}^x \frac{3t}{1+t^6}\,dt$.

O domínio de F é $\mathbb{R}$, pois $f(t) = \dfrac{3t}{1+t^6}$ é contínua em $\mathbb{R}$.

Pelo teorema fundamental do cálculo, temos:

$$F'(x) = \left[\int_{-2}^{x} f(t)\, dt\right]' = f(x)$$

$$F'(x) = \frac{3x}{1+x^6}$$

Na notação de Leibniz, $\dfrac{d}{dx}\left(\displaystyle\int_{-2}^{x} \frac{3t}{1+t^6}\, dt\right) = \dfrac{3x}{1+x^6}$.

c) $F(x) = \displaystyle\int_{x}^{2} \cos t^4\, dt = -\int_{2}^{x} \cos t^4\, dt$

$$F'(x) = \left[-\int_{2}^{x} \cos t^4\, dt\right]' = -f(x)$$

$$F'(x) = -\cos x^4$$

e) $F(x) = \displaystyle\int_{0}^{2x} \cos t^2\, dt$

Seja $u = 2x$

$$\frac{dF}{dx} = \frac{d}{du}\left(\int_{0}^{u} \cos t^2\, dt\right)\frac{du}{dx} = f(u)\cdot\frac{du}{dx}$$

$$\frac{dF}{dx} = \cos u^2 \cdot 2$$

$$\frac{dF}{dx} = 2\cos 4x^2$$

$F'(x) = 2\cos 4x^2$**.**

f) $F(x) = \displaystyle\int_{x^2}^{x^3} \frac{1}{5+t^4}\, dt$

$$F(x) = \int_{x^2}^{1} \frac{1}{5+t^4}\, dt + \int_{1}^{x^3} \frac{1}{5+t^4}\, dt$$

$$F(x) = \int_{1}^{x^3} \frac{1}{5+t^4}\, dt - \int_{1}^{x^2} \frac{1}{5+t^4}\, dt$$

$$F'(x) = \frac{1}{5+(x^3)^4}\,(x^3)' - \frac{1}{5+(x^2)^4}\,(x^2)'$$

$$F'(x) = \frac{3x^2}{5+x^{12}} - \frac{2x}{5+x^8}.$$

De outra forma:

$$F(x) = G(x^3) - G(x^2)$$

$$F'(x) = G'(x^3)\cdot 3x^2 - G'(x^2)\cdot 2x \qquad \text{onde } G'(t) = \frac{1}{5+t^4}$$

$$F'(x) = \frac{3x^2}{5+x^{12}} - \frac{2x}{5+x^8}.$$

j) $F(x) = \int_0^x (x-t)\,e^{-t^2}\,dt = \underbrace{\int_0^x xe^{-t^2}\,dt}_{F_1(x)} - \underbrace{\int_0^x te^{-t^2}\,dt}_{F_2(x)}$

$$F_1(x) = \int_0^x x\cdot e^{-t^2}\,dt = x\int_0^x e^{-t^2}\,dt$$

$$F_1'(x) = \frac{d}{dx}\left(\int_0^x x\cdot e^{-t^2}\,dt\right) = xe^{-x^2} + \int_0^x e^{-t^2}\,dt$$

$$F_2'(x) = xe^{-x^2}$$

$$F'(x) = F_1'(x) - F_2'(x) = xe^{-x^2} + \int_0^x e^{-t^2}\,dt - xe^{-x^2}$$

$$F'(x) = \int_0^x e^{-t^2}\,dt.$$

2. Seja $F(x) = \int_1^{x^3+3x^2} f(t)\,dt$

$$F'(x) = \frac{d}{du}\left(\int_1^u f(t)\,dt\right)\frac{du}{dx} = f(u)\,\frac{du}{dx} \quad \text{onde } u = x^3+3x^2.$$

$$F'(x) = f(x^3+3x^2)\cdot(3x^2+6x).$$

Supondo $f(t) \geqslant 0$ e contínua em $\mathbb{R}$, temos $f(x^3+3x^2) \geqslant 0$ em $\mathbb{R}$.
O sinal de $F'(x)$ depende de $3x^2+6x$.

Assim: $3x^2+6x \geqslant 0$ em $]-\infty, -2]$ e $[0, +\infty[$.
$3x^2+6x \leqslant 0$ em $[-2, 0]$.

Daí,

$F'(x) \geqslant 0$ em $]-\infty, -2]$ e $[0, +\infty[$ $\Rightarrow$ $F(x)$ é crescente em $]-\infty, -2]$ e $[0, +\infty[$.
$F'(x) \leqslant 0$ em $[-2, 0]$ $\Rightarrow$ $F(x)$ é decrescente em $[-2, 0]$.

3. $\varphi(x) = 1 + \int_0^x t\, \varphi(t)\, dt$ ①

$$\varphi'(x) = x\, \varphi(x) \quad \Rightarrow \quad \frac{d\varphi(x)}{dx} = x\, \varphi(x) \quad \Rightarrow \quad \frac{d\varphi(x)}{\varphi(x)} = x\, dx$$

Então, $\ln \varphi(x) = \dfrac{x^2}{2} + c$

$$\varphi(x) = e^{\frac{x^2}{2}} \cdot e^c \quad ②$$

De ① vem $\varphi(0) = 1$. Comparando com ②: $\varphi(0) = e^c$. Temos $e^c = 1$. Logo, $\varphi(x) = e^{\frac{x^2}{2}}$

6. Seja $F(x) = \int_1^x e^{-t^2}\, dt$

Para calcular $\int_0^1 F(x)\, dx$ vamos integrar por partes, considerando $f(x) = F(x)$ e $g'(x) = 1$. Então,

$$\int_0^1 F(x)\, dx = [x\, F(x)]_0^1 - \int_0^1 x\, F'(x)\, dx$$

$$\int_0^1 F(x)\, dx = \left[x \int_1^x e^{-t^2}\, dt \right]_0^1 - \int_0^1 x\, F'(x)\, dx$$

Temos $\left[x \int_1^x e^{-t^2}\, dt \right]_0^1 = 0$ $\left(\text{pois} \int_1^1 e^{-t^2}\, dt = 0 \text{ e } x \int_1^x e^{-t^2}\, dt = 0 \text{ se } x = 0 \right)$.

Como $F'(x) = e^{-x^2}$, segue que

$$\int_0^1 F(x)\, dx = -\int_0^1 x\, e^{-x^2}\, dx = \left[\frac{e^{-x^2}}{2} \right]_0^1 = \frac{1}{2e} - \frac{1}{2}$$

7. $G(x) = \int_\pi^x \operatorname{sen} t^2\, dt$

$$G'(x) = \left(\int_\pi^x \operatorname{sen} t^2\, dt \right)' = \operatorname{sen} x^2$$

Para calcular $\int_0^{\pi} G(x)\, dx$ vamos integrar por partes:

$$\begin{aligned}
\int_0^{\pi} G(x)\, dx &= \left[x\, G(x)\right]_0^{\pi} - \int_0^{\pi} x\, G'(x)\, dx \\
&= \underbrace{\left[x \int_{\pi}^{x} \operatorname{sen} t^2\, dt\right]_0^{\pi}} - \int_0^{\pi} x \cdot \operatorname{sen} x^2\, dx \\
&= \qquad\quad 0 \qquad\quad - \left(-\frac{1}{2}\right)\left[\cos x^2\right]_0^{\pi}
\end{aligned}$$

Logo, $\int_0^{\pi} G(x)\, dx = \frac{1}{2}\left[\cos \pi^2 - 1\right]$.