CAPÍTULO 3

Exercícios 3.1

1.

***a*)** $\displaystyle\int_1^{\infty} \frac{1}{x^3}\,dx = \lim_{t\to\infty}\int_1^{t} \frac{1}{x^3}\,dx = \lim_{t\to\infty}\left\{\left[-\frac{x^{-2}}{2}\right]_1^t\right\}$

$$= \lim_{t\to\infty}\left(-\frac{1}{2t^2}+\frac{1}{2}\right) = \frac{1}{2}.$$

***c*)** $\displaystyle\int_0^{\infty} e^{-sx}\,dx = \lim_{t\to\infty}\int_0^{t} e^{-sx}\,dx = \lim_{t\to\infty}\left\{\left[-\frac{e^{-sx}}{s}\right]_0^t\right\}$

$$= \lim_{t\to\infty}\left[-\frac{1}{se^{st}}+\frac{1}{s}\right] = \frac{1}{s},\ \text{pois}\ \lim_{t\to\infty}\frac{e^{-st}}{s} = 0.$$

***d*)** $\displaystyle\int_1^{\infty} \frac{1}{\sqrt{x}}\,dx = \lim_{t\to\infty}\int_1^{t} x^{-\frac{1}{2}}\,dx = \lim_{t\to\infty}\left\{\left[2\sqrt{x}\right]_1^t\right\}$

$$= \lim_{t\to\infty} = \left[2\sqrt{x}-2\right] = \infty.$$

***e*)** $\displaystyle\int_1^{\infty} te^{-t}\,dt = \lim_{s\to\infty}\int_0^{s} te^{-t}\,dt$

$$= \lim_{s\to\infty}\left[te^{-t}+\int_0^{s} e^{-t}\,dt\right]_0^s = \lim_{s\to\infty}\left[se^{-s}-e^{-s}+1\right] = 1,$$

pois $\displaystyle\lim_{s\to\infty} se^{-s} = 0$ e $\displaystyle\lim_{s\to\infty} e^{-s} = 0$

***n*)** $\displaystyle\int_0^{\infty} \frac{x}{1+x^4}\,dx = \lim_{t\to\infty}\int_0^{t} \frac{x}{1+x^4}\,dx.$

Façamos a mudança de variável

$x^2 = \text{tg}\, y \Rightarrow 2x\,dx = \sec^2 y\,dy \Rightarrow x\,dx = \dfrac{\sec^2 y\,dy}{2}$

$1 + x^4 = 1 + \text{tg}^2\, y = \sec^2 y$

$x = 0 \quad \Rightarrow \quad y = 0$
$x = t \quad \Rightarrow \quad y = \text{arctg}\, t^2$

Logo, $\displaystyle\int_0^t \frac{x}{1+x^4}\, dx = \int_0^{\text{arctg}\, t^2} \frac{\sec^2 y\, dy}{2\sec^2 y} = \left[\frac{1}{2}\, y\right]_0^{\text{arctg}\, t^2}$
$\displaystyle = \frac{1}{2}\ \text{arctg}\, t^2.$

Portanto,

$$\int_0^\infty \frac{x}{1+x^4}\, dx = \lim_{t\to\infty} \frac{1}{2}\ \text{arctg}\, t^2 = \frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{4}.$$

q) $\displaystyle\int_1^\infty \frac{1}{x^3+x}\, dx = \int_1^\infty \frac{dx}{x(x^2+1)} = \lim_{t\to\infty}\int_1^t \frac{dx}{x(x^2+1)}$

$$= \lim_{t\to\infty}\left[\int_1^t \frac{dx}{x} - \int_1^t \frac{x\,dx}{x^2+1}\right] = \lim_{t\to\infty}\left\{[\ln x]_1^t - \left[\ln\sqrt{x^2+1}\right]_1^t\right\}$$

$$= \lim_{t\to\infty}\left\{\ln t - \ln\sqrt{t^2+1} + \ln\sqrt{2}\right\} = \lim_{t\to\infty}\left(\ln\frac{t}{\sqrt{t^2+1}} + \ln\sqrt{2}\right)$$

$$= \ln 1 + \ln\sqrt{2} = \ln\sqrt{2} \qquad \left(\text{pois } \ln \lim_{t\to\infty}\frac{t}{\sqrt{t^2+1}} = \ln 1 = 0\right).$$

2. $\displaystyle\int_1^\infty \frac{1}{x^\alpha}\, dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^\alpha}\, dx = \lim_{t\to\infty}\left[\frac{t^{1-\alpha}}{1-\alpha} - \frac{1}{1-\alpha}\right] (\alpha \neq 1).$

Se $1 - \alpha = 0$, temos $\displaystyle\int_1^\infty \frac{dx}{x} = \lim_{t\to\infty}\int_1^t \frac{dx}{x} = \lim_{t\to\infty} \ln t = \infty.$
$(\alpha = 1)$

Se $(1-\alpha) < 0$, temos $\displaystyle\lim_{t\to\infty}\int_1^t \frac{dx}{x^\alpha} = \lim_{t\to\infty}\left[\cancelto{0}{\frac{1}{(1-\alpha)t^{|1-\alpha|}}} - \frac{1}{1-\alpha}\right] = \frac{1}{\alpha-1}.$
$(\alpha > 1)$

Se $(1-\alpha) > 0$, temos $\displaystyle\lim_{t\to\infty}\int_1^t \frac{dx}{x^\alpha} = \lim_{t\to\infty}\left[\frac{t^{1-\alpha}}{1-\alpha} - \frac{1}{1-\alpha}\right] = \infty.$
$(\alpha < 1)$

Portanto,

$$\int_1^\infty \frac{1}{x^\alpha}\, dx = \begin{cases} \dfrac{1}{\alpha-1} & \text{se } \alpha > 1 \\ \infty & \text{se } \alpha \leqslant 1. \end{cases}$$

3.

***b*)** $\displaystyle\int_{-\infty}^{-1} \frac{1}{x^5}\,dx = \lim_{t\to-\infty} \int_t^{-1} x^{-5}\,dx$

$$= \lim_{t\to-\infty}\left[-\frac{1}{4}+\frac{1}{4t^4}\right] = -\frac{1}{4}.$$

***c*)** $\displaystyle\int_{-\infty}^{-1} \frac{1}{\sqrt[3]{x}}\,dx = \lim_{t\to-\infty}\int_t^{-1} x^{-\frac{1}{3}}\,dx = \lim_{t\to-\infty}\left\{\left[\frac{3}{2}x^{\frac{2}{3}}\right]_t^{-1}\right\} = \lim_{t\to-\infty}\left[\frac{3}{2}-\frac{3}{2}t^{\frac{2}{3}}\right] = -\infty.$

***h*)** Temos

$$f(x) = \begin{cases} \dfrac{1}{x^2} & \text{se } x < -1 \\ 1 & \text{se } -1 \leqslant x \leqslant 1 \\ \dfrac{1}{x^2} & \text{se } x > 1 \end{cases}$$

Então,

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{-1}\frac{1}{x^2}\,dx + \int_{-1}^{1} dx + \int_1^{\infty}\frac{1}{x^2}\,dx$$

$$= \lim_{t\to-\infty}\left(1+\frac{1}{t}\right) + 2 + \lim_{t\to\infty}\left(-\frac{1}{t}+1\right) = 1+2+1 = 4.$$

4. Temos

$$f(x) = \begin{cases} m & \text{se } -3 \leqslant x \leqslant 3 \\ 0 & \text{se } x < -3 \\ 0 & \text{se } x > 3 \end{cases}$$

Então,

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-3}^{3} m\,dx = [mx]_{-3}^{3} = 3m + 3m = 6m.$$

De $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = 1$, segue $6m = 1$ ou $m = \dfrac{1}{6}$.

5. $\displaystyle\int_{-\infty}^{\infty} e^{k|t|}\,dt = \int_{-\infty}^{0} e^{k|t|}\,dt + \int_0^{\infty} e^{k|t|}\,dt$ ①

$$\int_{-\infty}^{0} e^{k|t|}\,dt = \lim_{s\to-\infty}\int_s^0 e^{-kt}\,dt = \lim_{s\to-\infty}\left\{\left[-\frac{e^{-kt}}{k}\right]_s^0\right\}$$

$$= \lim_{s\to-\infty}\left[-\frac{1}{k}+\frac{e^{-ks}}{k}\right] = -\frac{1}{k} \quad (\text{se } k<0) \quad ②$$

$\dfrac{e^{-ks}}{k} \to 0$ (se $k < 0$) (se $k \geqslant 0$ a integral diverge)

$$\int_0^{\infty} e^{k|t|}\,dt = \lim_{s\to\infty}\int_0^{s} e^{kt}\,dt = \lim_{s\to\infty}\left\{\left[\frac{e^{kt}}{k}\right]_0^{s}\right\}$$

$$= \lim_{s\to\infty}\left[\frac{e^{ks}}{k} - \frac{1}{k}\right] = -\frac{1}{k} \quad (\text{se } k < 0) \quad ③$$

$$\frac{e^{ks}}{k} \to 0 \ (\text{se } k < 0)$$

Portanto, substituindo ② e ③ em ①:

$$\int_{-\infty}^{\infty} e^{k|t|}\,dt = -\frac{1}{k} - \frac{1}{k} \quad \Rightarrow \quad -\frac{2}{k} = 1 \quad \Rightarrow \; k = -2.$$

7.

a) $\displaystyle\int_0^{\infty} e^{-st}\, t^n\,dt = \lim_{u\to\infty}\int_0^{u} e^{-st}\, t^n\,dt$

Integrando por partes (considerando $f(t) = t^n$ e $g'(t) = e^{-st}$)

$$\lim_{u\to\infty}\int_0^{u} \underset{\substack{\uparrow\\ f}}{t^n}\, \underset{\substack{\uparrow\\ g'}}{e^{-st}}\,dt = \lim_{u\to\infty}\left\{\left[-\frac{1}{s} t^n\, e^{-st}\right]_0^{u} + \frac{n}{s}\int_0^{u} t^{n-1}\, e^{-st}\,dt\right\}$$

$$= \lim_{u\to\infty}\left[-\frac{1}{s} u^n\, e^{-su}\right] + \frac{n}{s}\int_0^{\infty} t^{n-1}\, e^{-st}\,dt.$$

$$-\frac{1}{s} u^n\, e^{-su} \to 0$$

Portanto,

$$\int_0^{\infty} e^{-st}\, t^n\,dt = \frac{n}{s}\int_0^{\infty} e^{-st}\, t^{n-1}\,dt.$$

b) Consideremos $\displaystyle\int_0^{\infty} t^{n-1}\, e^{-st}\,dt$ obtida em (a).

Integrando novamente por partes.

$$\int_0^{u} \underset{\substack{\uparrow\\ f}}{t^{n-1}}\, \underset{\substack{\uparrow\\ g'}}{e^{-st}}\,dt = \left[-\frac{t^{n-1}}{s}\, e^{-st}\right] + \frac{n-1}{s}\int_0^{u} t^{n-2}\, e^{-st}\,dt.$$

Daí, $\displaystyle\lim_{u\to\infty}\int_0^{u} t^{n-1}\, e^{-st}\,dt = \frac{n-1}{s}\int_0^{\infty} t^{n-2}\, e^{-st}\,dt.$

Portanto,

$$\int_0^{\infty} e^{-st}\, t^n\,dt = \frac{n\,(n-1)}{s^2}\int_0^{\infty} e^{-st}\, t^{n-2}\,dt.$$

Integrando n vezes por partes, temos:

$$\int_0^\infty e^{-st}\ t^n\ dt = \frac{n\,(n-1)(n-2)\cdot \ldots \cdot 1}{s^n} \cdot \int_0^\infty e^{-st}\ dt \cdot$$

Mas

$$\int_0^\infty e^{-st}\ dt = \lim_{u\to\infty} \int_0^u e^{-st}\ dt = \lim_{u\to\infty} \left[-\frac{e^{-st}}{s}\right]_0^u = \lim_{u\to\infty} \left[\frac{e^{-su}}{s} + \frac{1}{s}\right] = \frac{1}{s}.$$

Então,

$$\int_0^\infty e^{-st}\ t^n\ dt = \frac{n!}{s^n} \cdot \frac{1}{s} = \frac{n!}{s^{n+1}}.$$

8.

a) $\displaystyle\int_0^\infty e^{-st}\ \text{sen}\ \alpha t\ dt = \lim_{u\to\infty} \int_0^u e^{-st}\ \text{sen}\ \alpha t\ dt$

$$\int_0^u \underset{\substack{\uparrow \\ f}}{e^{-st}}\ \underset{\substack{\uparrow \\ g'}}{\text{sen}\ \alpha t}\, dt = \left[-e^{-st}\ \frac{\cos \alpha t}{\alpha}\right]_0^u - \int_0^u \frac{s}{\alpha} \cos \alpha t\ e^{-st}\ dt$$

$$= -\frac{e^{-su}}{\alpha} \cos \alpha u + \frac{1}{\alpha} - \frac{s}{\alpha} \int_0^u e^{-st} \cos \alpha t\ dt.$$

Assim,

$$\int_0^u e^{-st}\ \text{sen}\ \alpha t\ dt = -\frac{e^{su}}{\alpha} \cos \alpha u + \frac{1}{\alpha} - \frac{s}{\alpha} \int_0^u e^{-st} \cos \alpha t\ dt. \quad ①$$

Por outro lado,

$$\int_0^u \underset{\substack{\uparrow \\ f}}{e^{-st}}\ \underset{\substack{\uparrow \\ g'}}{\cos \alpha t}\, dt = \left[\frac{e^{-st}}{\alpha}\ \text{sen}\ \alpha t\right]_0^u + \int_0^u \frac{s}{\alpha}\ e^{-st}\ \text{sen}\ \alpha t\ dt$$

$$= \frac{e^{-su}}{a}\ \text{sen}\ \alpha u + \frac{s}{\alpha} \int_0^u e^{-st}\ \text{sen}\ \alpha t\ dt.$$

Então, $\displaystyle\int_0^u e^{-st} \cos \alpha t\ dt = \frac{e^{-su}}{\alpha}\ \text{sen}\ \alpha t + \frac{s}{\alpha} \int_0^u e^{-st}\ \text{sen}\ \alpha t\ dt \quad ②$

Substituindo ② em ①:

$$\int_0^u e^{-st}\ \text{sen}\ \alpha t\ dt = -\frac{e^{-su}}{\alpha} \cos \alpha u + \frac{1}{\alpha} - \frac{s}{\alpha^2}\ e^{-su}\ \text{sen}\ \alpha u - \frac{s^2}{\alpha^2} \int_0^u e^{-st}\ \text{sen}\ \alpha t\ dt$$

$$\underbrace{\left(1 + \frac{s^2}{\alpha^2}\right)}_{\left(\frac{\alpha^2 + s^2}{\alpha^2}\right)} \int_0^u e^{-st}\ \text{sen}\ \alpha t\ dt = -\frac{e^{-su}}{\alpha} \cos \alpha u + \frac{1}{\alpha} - \frac{s}{\alpha^2}\ e^{-su}\ \text{sen}\ \alpha u.$$

Sendo sen αu e cos αu limitadas e $\lim_{u\to\infty} \frac{e^{-su}}{\alpha} = 0 \quad (s > 0)$ resulta

$$\lim_{u\to\infty} \frac{e^{-su}}{\alpha} \text{ sen } \alpha u = 0 \text{ e } \lim_{u\to\infty} \frac{e^{-su}}{\alpha} \cos \alpha u = 0.$$

Daí,

$$\int_0^{\infty} e^{-st} \text{ sen } \alpha t\, dt = \lim_{u\to\infty} \int_0^u e^{-st} \text{ sen } \alpha t\, dt = \left[\lim_{u\to\infty}\left(-\frac{e^{-su}}{\alpha}\cos \alpha u\right)\right.$$

$$\left. + \lim_{u\to\infty} \frac{1}{\alpha} - \lim_{u\to\infty} \frac{s}{\alpha^2} e^{-su} \text{ sen } \alpha u\right] \cdot \left(\frac{\alpha^2}{s^2+\alpha^2}\right)$$

$$\int_0^{\infty} e^{-st} \text{ sen } \alpha t\, dt = \left(\frac{1}{\alpha}\right)\left(\frac{\alpha^2}{s^2+\alpha^2}\right) = \frac{\alpha}{s^2+\alpha^2}.$$

c) $\int_0^{\infty} e^{-st} e^{\alpha t}\, dt = \lim_{u\to\infty} \int_0^u e^{-(s-\alpha)t}\, dt$

$$= \lim_{u\to\infty} \left[-\frac{e^{-(s-\alpha)t}}{(s-\alpha)}\right]_0^u = \lim_{u\to\infty} \left[-\frac{e^{-(s-\alpha)u}}{(s-\alpha)} + \frac{1}{s-\alpha}\right]$$

$$\lim_{u\to\infty} \frac{e^{-(s-\alpha)t}}{s-\alpha} = 0, \quad \text{pois } s > \alpha$$

Então,

$$\int_0^{\infty} e^{-st}\, e^{\alpha t}\, dt = \frac{1}{s-\alpha}$$

9.

a) $f(t) = \text{sen } t + 3 \cos 2t$

$$\int_0^{\infty} e^{-st}\, f(t)\, dt = \int_0^{\infty} e^{-st}\, (\text{sen } t + 3\cos 2t)\, dt$$

$$= \int_0^{\infty} e^{-st} \text{ sen } t\, dt + 3\int_0^{\infty} e^{-st} \cos 2t\, dt.$$

De 8.*a*, resulta

$$\int_0^{\infty} e^{-st} \text{ sen } t\, dt = \frac{1}{s^2+1}\cdot$$

De 8.*b*, resulta

$$\int_0^{\infty} e^{-st} \cos 2t\, dt = \frac{s}{s^2+4}$$

Logo,

$$\int_0^{\infty} e^{-st}\ (\text{sen } t + 3\cos 2t)\ dt = \frac{1}{s^2+1} + \frac{3s}{s^2+4}\cdot$$

Exercícios 3.3

1.

a) $\displaystyle\int_0^1 \frac{1}{\sqrt[3]{x}}\, dx$

$f(x) = \dfrac{1}{\sqrt[3]{x}}$ é não-limitada em]0, 1] e integrável (segundo Riemann) em [t, 1] para $0 < t < 1$.

Portanto,

$$\int_0^1 \frac{1}{\sqrt[3]{x}}\, dx = \lim_{t\to 0^+} \int_t^1 \frac{1}{\sqrt[3]{x}}\, dx = \lim_{t\to 0^+} \left[\frac{3}{2}\, x^{\frac{2}{3}}\right]_t^1 = \lim_{t\to 0^+} \left[\frac{3}{2} - \frac{3}{2}\, t^{\frac{2}{3}}\right] = \frac{3}{2}.$$

c) $\displaystyle\int_1^3 \frac{x^2}{\sqrt{x^3-1}}\, dx$ (A função integranda é não-limitada em]1, 3].)

$$\int_1^3 \frac{x^2}{\sqrt{x^3-1}}\, dx = \lim_{t\to 1^+} \int_t^3 \frac{x^2}{(x^3-1)^{\frac{1}{2}}}\, dx = \lim_{t\to 1^+} \left[\frac{2}{3}(x^3-1)^{\frac{1}{2}}\right]_t^3$$
$$= \frac{2}{3}\sqrt{26}.$$

3.

a) $\displaystyle\int_0^1 \frac{1}{\sqrt{1-x^2}}\, dx$ (A função integranda é não-limitada em [0, 1[.)

$$\int_0^1 \frac{1}{\sqrt{1-x^2}}\, dx = \lim_{t\to 1^-} \int_0^t \frac{dx}{\sqrt{1-x^2}} = \lim_{t\to 1^-} \left[\text{arcsen } x\right]_0^t$$
$$= \frac{\pi}{2}.$$

Exercícios 3.4

1.

a) Para $x \geqslant 1$, temos $\dfrac{1}{x^5+3x+1} \leqslant \dfrac{1}{x^5}$.

Segue, pelo critério de comparação, que $\displaystyle\int_1^{\infty} \frac{dx}{x^5+3x+1}$ é convergente, pois $\displaystyle\int_1^{\infty} \frac{dx}{x^5}$ é convergente.

c) Para $x \geqslant 2$, temos $\dfrac{1}{\sqrt[3]{x^4 + 2x + 1}} \leqslant \dfrac{1}{\sqrt[3]{x^4}}$.

Como $\displaystyle\int_2^{\infty} \frac{dx}{\sqrt[3]{x^4}}$ converge, segue, pelo critério de comparação, que $\displaystyle\int_2^{\infty} \frac{dx}{\sqrt{x^4 + 2x + 1}}$ converge.

e) Temos $0 \leqslant \left| \dfrac{\cos 3x}{x^3} \right| \leqslant \dfrac{1}{x^3}$ (para $x \geqslant 1$).

Como $\displaystyle\int_1^{\infty} \frac{dx}{x^3}$ converge, segue, pelo critério de comparação, que $\displaystyle\int_1^{\infty} \left| \frac{\cos 3x}{x^3} \right| dx$ converge.

Pelo exemplo 3, $\displaystyle\int_1^{\infty} \frac{\cos 3x}{x^3}\, dx$ converge.

j) $0 \leqslant \dfrac{x}{\sqrt{x^2 + x + 1}} \cdot e^{-x} \leqslant e^{-x}, x \geqslant 0.$

Como $\displaystyle\int_0^{\infty} e^{-x}\, dx$ é convergente, pelo critério de comparação $\displaystyle\int_0^{\infty} \frac{xe^{-x}}{\sqrt{x^2 + x + 1}}\, dx$ converge.

m) $\displaystyle\int_{-\infty}^{\infty} \frac{1}{x^4 + x^2 + 1}\, dx = 2\int_0^{\infty} \frac{1}{x^4 + x^2 + 1}\, dx$, pois o integrando é função par.

$\displaystyle\int_0^{\infty} \frac{1}{x^4 + x^2 + 1}\, dx = \int_0^{1} \frac{1}{x^4 + x^2 + 1}\, dx + \int_1^{\infty} \frac{1}{x^4 + x^2 + 1}\, dx.$ A convergência da última integral segue do critério de comparação, pois $\dfrac{1}{x^4 + x^2 + 1} \leqslant \dfrac{1}{x^4}$ para $x \geqslant 1$.

(Observe que $\displaystyle\int_0^{1} \frac{1}{x^4 + x^2 + 1}\, dx$ existe, pois a integranda é contínua em [0, 1].)

Portanto, $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^4 + x^2 + 1}\, dx$ converge.

2. Da hipótese, existe $b > 0$ tal que $\dfrac{L}{2} \leqslant x^{\alpha}\, f(x) \leqslant \dfrac{3L}{2}$, para $x \geqslant b$. Daí,

$\dfrac{L}{2x^{\alpha}} \leqslant f(x) \leqslant \dfrac{3L}{2x^{\alpha}}$, para $x \geqslant b$. Sendo $f(x)$ integrável em $[a, t]$, para $t \geqslant a$, temos

$\displaystyle\int_a^{\infty} f(x)\, dx = \int_a^{b} f(x)\, dx + \int_b^{\infty} f(x)\, dx.$ Já sabemos que $\displaystyle\int_b^{\infty} \frac{1}{x^{\alpha}}\, dx$, $b > 0$, é convergente para $\alpha > 1$ e divergente para $\alpha \leqslant 1$. Pelo critério de comparação, temos:

a) $\alpha > 1 \Rightarrow \int_b^{\infty} f(x)\,dx$ convergente $\Rightarrow \int_a^{\infty} f(x)\,dx$ convergente;

b) $\alpha \leqslant 1 \Rightarrow \int_b^{\infty} f(x)\,dx$ divergente $\Rightarrow \int_a^{\infty} f(x)\,dx$ divergente.

3.

a) $\int_2^{\infty} \frac{x^6 - x + 1}{x^7 - 2x^2 + 3}\,dx$

Seja $\underbrace{\frac{x^6 - x + 1}{x^7 - 2x^2 + 3}}_{f(x)} = \frac{1}{x} \underbrace{\left(\frac{1 - \frac{1}{x^5} - \frac{1}{x^6}}{1 - \frac{2}{x^5} + \frac{3}{x^7}} \right)}_{g(x)}$

Logo, $f(x) = \frac{1}{x}\,g(x)$, $f(x) \geqslant 0$ em $[2, +\infty[$ e $\lim_{x \to \infty} g(x) = 1$.

Por (2),

$\alpha = 1 \;\Rightarrow\; \int_2^{\infty} \frac{x^6 - x + 1}{x^7 - 2x^2 + 3}\,dx$ é divergente.

b) $\int_{10}^{\infty} \frac{x^5 - 3}{\sqrt{x^{20} + x^{10} - 1}}\,dx$

Temos $\underbrace{\frac{x^5 - 3}{\sqrt{x^{20} + x^{10} - 1}}}_{f(x)} = \frac{1}{x^5} \underbrace{\frac{1 - \frac{3}{x^5}}{\sqrt{1 + \frac{1}{x^{10}} - \frac{1}{x^{20}}}}}_{g(x)}.$

Assim $f(x) = \frac{1}{x^5}\,g(x)$ com $\lim_{x \to \infty} g(x) = 1.$

Por (2),

$\alpha = 5 \;\Rightarrow\; \alpha > 1 \;\Rightarrow\; \int_{10}^{\infty} \frac{x^5 - 3}{\sqrt{x^{20} + x^{10} - 1}}\,dx$ converge.

d) $\int_1^{\infty} \frac{\ln x}{x \ln(x+1)}\,dx$

Seja $\underbrace{\frac{\ln x}{x \ln(x+1)}}_{f(x)} = \frac{1}{x} \underbrace{\frac{\ln x}{\ln(x+1)}}_{g(x)}$. Temos $\lim_{x \to \infty} \underbrace{\frac{\ln x}{\ln(x+1)}}_{g(x)} = \lim_{x \to \infty} \frac{\frac{1}{x}}{\frac{1}{x+1}}.$

Então, $\lim\limits_{x\to\infty} g(x) = \lim\limits_{x\to\infty} \dfrac{x+1}{x} = 1.$

Por (2), $\alpha = 1 \;\Rightarrow\; \displaystyle\int_1^{\infty} \frac{\ln x}{x\,\ln(x+1)}\,dx$ diverge.

5. Integrando por partes, $\displaystyle\int_0^u e^{-st} f'(t)\,dt = e^{-su} f(u) - f(0) + s\int_0^u e^{-st} f(t)\,dt.$ Sendo f de ordem exponencial, existem $\gamma > 0$ e $M > 0$ tais que, para $t \geqslant 0$, $|e^{-su} f(u)| \leqslant Me^{-(s-\gamma)u}$. Daí, para $s > \gamma$, $\lim\limits_{u\to\infty} e^{-su} f(u) = 0$ e da convergência da integral $\displaystyle\int_0^{\infty} e^{-(s-\gamma)t}\,dt$, segue a convergência de $\displaystyle\int_0^{\infty} e^{-st} f(t)\,dt.$ Portanto, $\displaystyle\int_0^{\infty} e^{-st} f'(t)\,dt$ é convergente e $\displaystyle\int_0^{\infty} e^{-st} f'(t)\,dt = s\int_0^{u} e^{-st} f(t)\,dt - f(0).$

6. Seja $f'(t) + 3f(t) = t$, para todo t real.

Daí, $f'(t) = t - 3f(t)$ ①

Supondo f de ordem exponencial γ, temos, de (5), para todo $s > \gamma$,

$$\int_0^{\infty} e^{-st} f'(t)\,dt = s\int_0^{\infty} e^{-st} f(t)\,dt - f(0).$$

De ①: $\displaystyle\int_0^{\infty} e^{-st}\,(t - 3f(t))\,dt = s\int_0^{\infty} e^{-st} f(t)\,dt - f(0)$

$$\int_0^{\infty} te^{-st}\,dt - 3\int_0^{\infty} f(t)\,e^{-st}\,dt = s\int_0^{\infty} e^{-st} f(t)\,dt - f(0)$$

$$(s+3)\int_0^{\infty} e^{-st} f(t)\,dt = \int_0^{\infty} te^{-st}\,dt + f(0). \quad ②$$

Agora, $\displaystyle\int_0^{\infty} te^{-st}\,dt = \frac{1}{s^2}.$ ③ (Do Exercício 8, Seção 3.1.)

Substituindo ③ em ②:

$$\int_0^{\infty} e^{-st} f(t)\,dt = \frac{1}{s^2(s+3)} + \frac{f(0)}{(s+3)}$$

$$\int_0^{\infty} e^{-st} f(t)\,dt = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+3} + \frac{f(0)}{s+3}$$

De $\dfrac{A}{s} + \dfrac{B}{s^2} + \dfrac{C}{s+3} = \dfrac{1}{s^2(s+3)}$, segue:

$\dfrac{(A+C)\,s^2 + (3A+B)\,s + 3B}{s^2(s+3)} = \dfrac{1}{s^2(s+3)}.$ Donde $3B = 1$;

$B = \dfrac{1}{3}$; $3A + B = 0$; $A = -\dfrac{1}{9}$; $A + C = 0$; $C = \dfrac{1}{9}.$

Portanto,

$$\int_0^{\infty} e^{-st}\ f(t)\,dt = -\frac{1}{9}\cdot\frac{1}{s}+\frac{1}{3}\cdot\frac{1}{s^2}+\frac{1}{9}\cdot\frac{1}{s+3}+\frac{f(0)}{s+3}.$$

Supondo $f(0) = 1$, temos

$$\int_0^{\infty} e^{-st}\ f(t)\,dt = \frac{10}{9}\cdot\frac{1}{s+3}-\frac{1}{9}\cdot\frac{1}{s}+\frac{1}{3}\cdot\frac{1}{s^2}.$$

Utilizando o Exercício 8, Seção 3.1, resulta:

$$f(t)=\frac{10}{9}\,e^{-3t}-\frac{1}{9}+\frac{1}{3}\,t.$$

7.

a) $f'(t) - 2f(t) = \cos t$ e $f(0) = 2$

$$\int_0^{\infty} e^{-st}\ f'(t)\,dt = s\int_0^{\infty} e^{-st}\ f(t)\,dt - f(0)$$

$$\int_0^{\infty} e^{-st}\left[2f(t)+\cos t\right]dt = s\int_0^{\infty} e^{-st}\ f(t)\,dt - 2$$

$$(s-2)\int_0^{\infty} e^{-st}\ f(t)\,dt = \int_0^{\infty} e^{-st}\cos t\,dt + 2$$

$$\int_0^{\infty} e^{-st}\ f(t)\,dt = \frac{1}{s-2}\underbrace{\int_0^{\infty} e^{-st}\cos t\,dt}_{\frac{s}{s^2+1}} + \frac{2}{s-2}$$

(do Exercício 8, Seção 3.1)

$$\int_0^{\infty} e^{-st}\ f(t)\,dt = \frac{s}{(s-2)(s^2+1)}+\frac{2}{s-2}$$

$$\int_0^{\infty} e^{-st}\ f(t)\,dt = \frac{A}{s-2}+\frac{Bs+C}{s^2+1}+\frac{2}{s-2}$$

$$\frac{(A+B)s^2-(2B-C)s+A-2C}{(s-2)(s^2+1)} = \frac{s}{(s-2)(s^2+1)}$$

De $\begin{cases} A+B=0 \\ 2B-C=-1 \\ A-2C=0 \end{cases}$ temos $A=\frac{2}{5}$; $B=-\frac{2}{5}$ e $C=\frac{1}{5}$

Portanto,

$$\int_0^{\infty} e^{-st}\ f(t)\,dt = \frac{2}{5}\cdot\frac{1}{s-2}+\frac{-\frac{2}{5}s+\frac{1}{5}}{s^2+1}+\frac{2}{s-2}$$

$$\int_0^{\infty} e^{-st}\ f(t)\,dt = \frac{12}{5}\cdot\frac{1}{s-2}-\frac{2}{5}\,\frac{s}{s^2+1}+\frac{1}{5}\cdot\frac{1}{s^2+1}$$

$$f(t)=\frac{12}{5}\,e^{2t}-\frac{2}{5}\cos t+\frac{1}{5}\,\text{sen}\,t.$$

b) $f'(t) + f(t) = e^{2t}$, $f(0) = -1$

$$f'(t) = e^{2t} - f(t)$$

$$\int_0^\infty e^{-st}\ f'(t) = s\int_0^\infty e^{-st}\ f(t)\,dt - f(0)$$

$$\int_0^\infty e^{-st}\left[e^{2t} - f(t)\right]dt = s\int_0^\infty e^{-st}\ f(t)\,dt - \underbrace{f(0)}_{(-1)}$$

$$(s+1)\int_0^\infty e^{-st}\ f(t)\,dt = \underbrace{\int_0^\infty e^{-st}\cdot e^{2t}\ dt}_{\frac{1}{s-2}\ \text{(do Exercício 8, Seção 3.1)}} - 1$$

$$\int_0^\infty e^{-st}\ f(t)\,dt = \frac{1}{(s+1)(s-2)} - \frac{1}{s+1}$$

$$\frac{A}{s+1} + \frac{B}{s-2} = \frac{1}{(s+1)(s-2)} \quad \Rightarrow \quad \frac{(A+B)s + (B-2A)}{(s+1)(s-2)} = \frac{1}{(s+1)(s-2)}$$

$$\Rightarrow A + B = 0 \ \text{ e } \ B - 2A = 1 \quad \Rightarrow \quad A = -\frac{1}{3} \text{ e } B = \frac{1}{3}$$

$$\int_0^\infty e^{-st}\ f(t)\,dt = -\frac{4}{3}\cdot\frac{1}{s+1} + \frac{1}{3}\cdot\frac{2}{s-2}$$

Portanto, utilizando o Exercício 8, Seção 3.1 temos

$$f(t) = -\frac{4}{3}\,e^{-t} + \frac{1}{3}\,e^{2t},$$

pois, $\displaystyle\int_0^\infty e^{-st}\ e^{\alpha t}\ dt = \frac{1}{s-\alpha} \qquad (s > \alpha).$