CAPÍTULO 4

Exercícios 4.1

1.
a) Dizemos que f é uma função densidade de probabilidade se
i) $f(x) \geqslant 0, \forall x$

ii) $\int_{-\infty}^{\infty} f(x)\, dx = 1.$

Seja $f(x) = k\, e^{-x^2}$ para $x \geqslant 0$ e $f(x) = 0$, para $x < 0$.
De *i* segue que $k > 0$.

Agora $\int_{-\infty}^{\infty} f(x)\, dx = \int_{0}^{\infty} f(x)\, dx = \int_{0}^{\infty} kxe^{-x^2}\, dx = \left[-\frac{ke^{-x^2}}{2}\right]_0^{\infty} = \frac{k}{2}$

De *ii* segue: $\frac{k}{2} = 1 \Rightarrow k = 2.$

c) De *i* segue que $kx\,(x - 5) \geqslant 0$.
Como $0 \leqslant x \leqslant 5$, temos $x - 5 \leqslant 0$ e $k \leqslant 0$.

De *ii* segue $\int_0^5 kx(x-5)\, dx = 1.$

Agora, $\int_0^5 kx(x-5)\, dx = \int_0^5 kx^2\, dx - \int_0^5 5k\, dx = k\left[\frac{x^3}{3}\right]_0^5 - 5k\left[\frac{x^2}{2}\right]_0^5$

Logo, $\frac{125k}{3} - \frac{125k}{2} = 6 \Rightarrow -125k = 6 \Rightarrow k = -\frac{6}{125}$

d) De *i*, como $1 + 4x^2 \geqslant 0$, devemos ter $k \geqslant 0$.

De *ii* vem $\int_{-\infty}^{\infty} \frac{k}{1+4x^2}\, dx = 1 \Rightarrow k\left[\text{arctg}\, 2x\right]_{-\infty}^{\infty} = 2.$

Mas $k\left[\text{arctg}\, 2x\right]_{-\infty}^{\infty} = k\pi$, logo $k\pi = 2$ e $k = \frac{2}{\pi}$.

2. ***a***)

$\int_{400}^{\infty} kx^{-2} = 1 \Rightarrow \left[-kx^{-1}\right]_{400}^{\infty} = 1 \Rightarrow \frac{k}{400} = \Rightarrow k = 400$

b) $\int_{400}^{1000} 400\, x^{-2} = -400\, x^{-1}\Big]_{400}^{1000} = -\frac{400}{1000} + 1 = 0,6$

d) $\int_{2000}^{5000} 400\, x^{-2} = -400\, x^{-1}\Big]_{2000}^{5000} = -\frac{400}{5000} + \frac{400}{2000} = \frac{3}{25}$

Logo, $\frac{3}{25} \cdot 3200 = 384.$

Exercícios 4.2

1.

a) De $F(x) = \int_{-\infty}^{x} f(t)\, dt$ (função de distribuição) segue que

$F(x) = 0$ se $x < 0$

$F(x) = \frac{x}{5}$ se $0 \leqslant x \leqslant 5$

$F(x) = 1$ se $x > 5$

Observamos que $\lim_{x \to -\infty} F(x) = 0$ e $\lim_{x \to +\infty} F(x) = 1$

c) Seja a função de densidade de probabilidade

$f(x) = \frac{1}{2}\, e^{-|x|}$ para todo x real.

Temos

$$f(x) = \begin{cases} \frac{1}{2} e^{x} & \text{se} \quad x \leqslant 0 \\ \frac{1}{2} e^{-x} & \text{se} \quad x > 0. \end{cases}$$

Logo,

$$F(x) = \int_{-\infty}^{x} \frac{1}{2}\, e^{t}\, dt = \frac{1}{2} \lim_{S \to -\infty} \int_{S}^{x} e^{t}\, dt = \frac{1}{2}\, e^{x} \text{ se } x \leqslant 0$$

e

$$F(x) = \int_{-\infty}^{0} \frac{1}{2}\, e^{t}\, dt + \int_{0}^{x} \frac{1}{2}\, e^{-t}\, dt = \frac{1}{2} - \frac{e^{-x}}{2} + \frac{1}{2} = 1 - \frac{e^{-x}}{2} \text{ se } x > 0.$$

Portanto,

$$F(x) = \begin{cases} \frac{1}{2} e^{x} & \text{se} \quad x \leqslant 0 \\ 1 - \frac{e^{-x}}{2} & \text{se} \quad x > 0. \end{cases}$$

2. $f(x) = F'(x) = \dfrac{d}{dx}\left[\dfrac{1}{\pi}\int_{-\infty}^{2x} \dfrac{1}{1+t^2}\, dt\right] = \dfrac{1}{\pi(1+4x^2)}\,(2x)' = \dfrac{2x}{\pi(1+4x^2)}.$

Então, $f(x) = \dfrac{2}{\pi(1+4x^2)}$ é a função densidade de probabilidade.

Exercícios 4.3

1.

a) $E(X) = \int_{-\infty}^{\infty} xf(x)\, dx.$

Sendo $f(x) = \begin{cases} \dfrac{1}{b-a} & \text{se } a \leqslant x \leqslant b \\ 0 & \text{se } x < a \text{ e } x > b \end{cases}$

Temos

$$E(X) = \int_a^b x \cdot \frac{1}{b-a}\, dx = \frac{1}{b-a} \cdot \left[\frac{x^2}{2}\right]_a^b = \frac{b^2 - a^2}{2(b-a)} = \frac{a+b}{2}$$

$$\text{Var}(X) = \int_a^b x^2 f(x)\, dx - [E(x)]^2 = \int_a^b \frac{x}{b-a}\, dx - \left(\frac{a+b}{2}\right)^2$$

$$\text{Var}(X) = \frac{1}{b-a} \cdot \left[\frac{x^3}{3}\right]_a^b - \left(\frac{a+b}{2}\right)^2 = \frac{1}{b-a} \cdot \frac{b^3 - a^3}{3} - \frac{(a+b)^2}{4}$$

$$\text{Var}(X) = \frac{(b-a)(b^2 + ab + a^2)}{3(b-a)} - \frac{(a+b)^2}{4} = \frac{4(b^2 + ab + a^2) - 3(a+b)^2}{12}$$

$$\text{Var}(X) = \frac{(b-a)^2}{12}$$

c) Seja a função densidade de probabilidade:

$$f(x) = \begin{cases} xe^{-x} & \text{se } x \geqslant 0 \\ 0 & \text{se } x < 0 \end{cases}$$

$$E(X) = \int_0^{\infty} x\, f(x)\, dx = \int_0^{\infty} x^2 e^{-x}\, dx$$

Integrando duas vezes por partes:

$$\int_0^s x^2 e^{-x}\, dx = \left[-x^2 e^{-x}\right]_0^s + 2\int_0^s xe^{-x}\, dx$$

$$= \left[x^2 e^{-x}\right]_0^s + 2\left[-xe^{-x} - e^{-x}\right]_0^s = \left[s^2 e^{-s}\right] + 2\left[-se^{-s} - e^{-s} + 1\right]$$

De $\lim\limits_{s\to\infty} s^2 e^{-s} = 0$; $\lim\limits_{s\to\infty} se^{-s} = 0$ e $\lim\limits_{s\to\infty} e^{-s} = 0$

resulta

$$E(X) = \int_0^{\infty} x^2 e^{-x}\, dx = 2$$

$$\text{Var}(X) = \int_0^{\infty} x^2\, f(x)\, dx - [E(X)]^2$$

$$\text{Var}(X) = \int_0^{\infty} x^3 e^{-x}\, dx - 4$$

Integrando quatro vezes por partes obtemos:

$$\int_0^{s} x^3\, e^{-x}\, dx = \left[-x^3 e^{-x}\right]_0^s + 3\int_0^{s} x^2 e^{-x}\, dx$$

De $\lim\limits_{s\to\infty} s^3 e^{-s} = 0$ e de $\lim\limits_{s\to\infty} \int_0^{s} x^2 e^{-x}\, dx = E(X) = 2$ resulta

$$\int_0^{\infty} x^3 e^{-x}\, dx = 6$$

Var (X) = 6 − 4 = 2.

Exercícios 4.4

1. Seja $X : N(\mu, \sigma^2)$ (isto é, a variável aleatória X tem distribuição normal, com média μ e variância σ^2).

Portanto, a sua função densidade de probabilidade é dada por:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}\; e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad x \text{ real.}$$

Temos, considerando $r > 0$ um número qualquer:

$$P(\mu - r\sigma \leqslant X \leqslant \mu + r\sigma) = \frac{1}{\sigma\sqrt{2\pi}} \int_{\mu - r\sigma}^{\mu + r\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx$$

Fazendo a mudança de variável

$$z = \frac{x-\mu}{\sigma} \quad \Rightarrow \quad dx = \sigma\, dz$$
$$x = \mu - r\sigma \quad \Rightarrow \quad z = -r$$
$$x = \mu + r\sigma \quad \Rightarrow \quad z = r$$

Logo,

$$P(\mu - r\sigma \leqslant X \leqslant \mu + r\sigma) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-r}^{r} e^{\frac{-z^2}{2}}\, \sigma\, dz, \text{ ou seja,}$$

$$P(\mu - r\sigma \leqslant X \leqslant \mu + r\sigma) = \frac{1}{\sqrt{2\pi}} \int_{-r}^{r} e^{\frac{-z^2}{2}}\, dz.$$

Logo, a probabilidade de X estar entre $\mu - r\sigma$ e $\mu + r\sigma$ só depende de r.

2. Seja $X : N(\mu, \sigma^2)$
Temos

$$P(a \leqslant X \leqslant b) = \frac{1}{\sigma\sqrt{2\pi}} \int_a^b e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx.$$

Fazendo a mudança de variável: $z = \dfrac{x-\mu}{\sigma}$, $dx = \sigma\, dz$

$x = a \Rightarrow z = \dfrac{a-\mu}{\sigma}$ e $x = b \Rightarrow z = \dfrac{b-\mu}{\sigma}$

Logo,

$$P(a \leqslant X \leqslant b) = \frac{1}{\sqrt{2\pi}} \int_{\frac{a-\mu}{\sigma}}^{\frac{b-\mu}{\sigma}} e^{-\frac{z^2}{2}} dz.$$

3. Sejam $X : N(50, 16)$ e $Y: N(60, 25)$

a) $P(X \leqslant x) = P(Y \leqslant x)$.

Temos $P(X \leqslant x) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\frac{x-50}{5}} e^{-\frac{z^2}{2}} dz$ (pois $\mu = 50$ e $\sigma = 4$) e

$P(Y \leqslant x) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\frac{x-60}{5}} e^{-\frac{z^2}{2}} dz$ (pois $\mu = 60$ e $\sigma = 5$).

Comparando, resulta:

$$\frac{x-50}{4} = \frac{x-60}{5} \quad \Rightarrow \quad x = 10$$

b) $P(X \leqslant x) < P(Y \leqslant x)$.

Temos

$$\frac{x-50}{4} < \frac{x-60}{5} \quad \Rightarrow \quad x < 10$$

5.

a) Seja $\varphi(\mu) = \dfrac{1}{\sigma\sqrt{2\pi}} \displaystyle\int_a^b e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx.$

Fazendo a mudança de variável

$$z = \frac{x-\mu}{\sigma} \quad \Rightarrow \quad dx = \sigma\, dz$$

$$x = a \quad \Rightarrow \quad z = \frac{a-\mu}{\sigma}$$

$$x = b \quad \Rightarrow \quad z = \frac{b-\mu}{\sigma}.$$

Portanto,

$$\varphi(\mu) = \frac{1}{\sigma\sqrt{2\pi}} \int_{\frac{a-\mu}{\sigma}}^{\frac{b-\mu}{\sigma}} e^{-\frac{z^2}{2}} \sigma \, dz$$

ou seja,

$$\varphi(\mu) = \frac{1}{\sqrt{2\pi}} \int_{\frac{a-\mu}{\sigma}}^{\frac{b-\mu}{\sigma}} e^{-\frac{z^2}{2}} dz.$$

b) Seja $f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$ (função contínua).

$$\frac{d\varphi}{d\mu} = f\left(\frac{b-\mu}{\sigma}\right) \frac{d}{d\mu}\left(\frac{b-\mu}{\sigma}\right) - f\left(\frac{a-\mu}{\sigma}\right) \frac{d}{d\mu}\left(\frac{a-\mu}{\sigma}\right)$$

$$\frac{d\varphi}{d\mu} = \frac{1}{\sqrt{2\pi}} \left[e^{-\frac{(b-\mu)^2}{2\sigma^2}} \left(-\frac{1}{\sigma}\right) - e^{-\frac{(a-\mu)^2}{2\sigma^2}} \left(-\frac{1}{\sigma}\right) \right]$$

$$\frac{d\varphi}{d\mu} = \left(\frac{1}{\sigma\sqrt{2\pi}}\right) \left[e^{-\frac{(a-\mu)^2}{2\sigma^2}} - e^{-\frac{(b-\mu)^2}{2\sigma^2}} \right].$$

De outra forma, para se obter $\frac{d\varphi}{d\mu}$, consideremos

$$\varphi(\mu) = \frac{1}{\sqrt{2\pi}} \int_{\frac{a-\mu}{\sigma}}^{\frac{b-\mu}{\sigma}} e^{-\frac{z^2}{2}} dz$$

$$\varphi(\mu) = \frac{1}{\sqrt{2\pi}} \left[\int_{\frac{a-\mu}{\sigma}}^{1} e^{-\frac{z^2}{2}} dz + \int_{1}^{\frac{b-\mu}{\sigma}} e^{-\frac{z^2}{2}} dz \right]$$

$$\varphi(\mu) = \frac{1}{\sqrt{2\pi}} \left[\int_{1}^{\frac{b-\mu}{\sigma}} e^{-\frac{z^2}{2}} dz - \int_{1}^{\frac{a-\mu}{\sigma}} e^{-\frac{z^2}{2}} dz \right]$$

$$\varphi'(\mu) = \frac{d\varphi}{d\mu} = \frac{1}{\sqrt{2\pi}} \left[e^{-\frac{(b-\mu)^2}{2\sigma^2}} \underbrace{\frac{d}{d\mu}\left(\frac{b-\mu}{\sigma}\right)}_{\left(-\frac{1}{\sigma}\right)} - e^{-\frac{(a-\mu)^2}{2\sigma^2}} \underbrace{\frac{d}{d\mu}\left(\frac{a-\mu}{\sigma}\right)}_{\left(-\frac{1}{\sigma}\right)} \right]$$

Então,

$$\frac{d\varphi}{d\mu} = \frac{1}{\sigma\sqrt{2\pi}} \left[e^{-\frac{(a-\mu)^2}{2\sigma^2}} - e^{-\frac{(b-\mu)^2}{2\sigma^2}} \right].$$

Exercícios 4.5

2. Seja $X : N(\mu, \sigma^2)$

$X = \ln Y$ (distribuição logonormal) $(Y > 0)$.

$$P(a < Y < b) = P(\ln a < X < \ln b) = \int_{\ln a}^{\ln b} f(x)\, dx.$$

Fazendo a mudança de variável $x = \ln y$ temos

$P(a < Y < b) = \displaystyle\int_a^b \frac{f(\ln y)\, dy}{y}$, para quaisquer a, b *reais* com $0 \leqslant a < b$.

Assim, a função densidade de probabilidade g da variável aleatória Y é dada por:

$$g(y) = \begin{cases} \dfrac{f(\ln y)}{y} & \text{se } y > 0 \\ 0 & \text{se } y \leqslant 0 \end{cases}$$

onde $f(x) = \dfrac{1}{\sigma\sqrt{2\pi}}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}$.

Exercícios 4.6

1. $\Gamma\left(\dfrac{1}{2}\right) = \displaystyle\int_0^\infty e^{-x} \cdot x^{-\frac{1}{2}}\, dx$

Fazendo $x = u^2$, $dx = 2u\, du$ temos

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-u^2}\, du = 2 \cdot \frac{\sqrt{\pi}}{2} = \sqrt{\pi}$$

$$\left(\text{pois } \int_0^\infty e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}\right).$$

3. Como $\Gamma(\alpha + 1) = \alpha\,\Gamma(\alpha)$ (Exemplo 4, *b*)

$$\Gamma\left(\frac{3}{2}\right) = \Gamma\left(\frac{1}{2}+1\right) = \frac{1}{2}\,\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi}.$$

$$\Gamma\left(\frac{5}{2}\right) = \Gamma\left(\frac{3}{2}+1\right) = \frac{3}{2}\,\Gamma\left(\frac{3}{2}\right) = \frac{3}{2}\left(\frac{1}{2}\sqrt{\pi}\right) = \frac{3}{4}\sqrt{\pi}.$$

4. $$\Gamma\left(\frac{2n+1}{2}\right) = \Gamma\left(\frac{2n-1}{2}+1\right) = \frac{2n-1}{2}\cdot\Gamma\left(\frac{2n-1}{2}\right)$$

$$= \frac{2n-1}{2}\,\Gamma\left(\frac{2n-3}{2}+1\right) = \frac{(2n-1)}{2}\cdot\frac{(2n-3)}{2}\cdot\Gamma\left(\frac{2n-3}{2}\right)$$

$$= \frac{(2n-1)}{2}\cdot\frac{(2n-3)}{2}\cdot\Gamma\left(\frac{2n-5}{2}+1\right) = \frac{(2n-1)}{2}\cdot\frac{(2n-3)}{2}\cdot\frac{(2n-5)}{2}\cdot\Gamma\left(\frac{2n-5}{2}\right)$$

$2(n-1)$ $2(n-2)$

$$= \ldots = \frac{(2n-1)}{2}\cdot\frac{(2n-3)}{2}\cdot\frac{(2n-5)}{2}\cdot\ldots\cdot\frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right)$$

$$= \frac{(2n-1)(2n-2)(2n-3)\,(2n-4)\,(2n-5)\cdot\ldots\cdot 3\cdot 1}{\underbrace{2\cdot 2\cdot 2\cdot\ldots\cdot 2}_{2^{2n-1}}\ \underbrace{(n-1)\,(n-2)\cdot\ldots\cdot 1}_{(n-1)!}}\sqrt{\pi}$$

$$\Gamma\left(\frac{2n+1}{2}\right) = \frac{(2n-1)!}{2^{2n-1}\,(n-1)!}\sqrt{\pi}.$$

Exercícios 4.7

3.

b) $$f(x) = \begin{cases} \beta x^{\beta-1}\ e^{-x^{\beta}} & \text{se } x > 0 \\ 0 & \text{se } x \leqslant 0. \end{cases}$$

$$E(X) = \int_0^{\infty} x\ \beta x^{\beta-1}\ e^{-x^{\beta}}\ dx.$$

Integrando por partes:

$$E(X) = \lim_{s\to\infty}\left\{\left[-x\cdot e^{-x^{\beta}}\right]_0^s + \int_0^s e^{-x^{\beta}}\ dx\right\}.$$

Como $\lim\limits_{s\to\infty} se^{-s^{\beta}} = 0$, resulta

$$E(X) = \int_0^{\infty} e^{-x^{\beta}}\ dx.$$

$$\text{Var}\,(X) = \int_0^{\infty} x^2\ f(x)\,dx - [E(X)]^2$$

$$\text{Var}\,(X) = \int_0^{\infty} x^2\ \beta x^{\beta-1} e^{-x^{\beta}}\ dx - [E(X)]^2$$

Integrando por partes:

$$\text{Var}\,(X) = \lim_{s\to\infty} \left\{ \left[-x^2 e^{-x^\beta} \right]_0^s + 2\int_0^s e^{-x^\beta}\, x\, dx \right\} - [E(X)]^2$$

Como $\lim\limits_{s\to\infty} -s^2 e^{-s^\beta} = 0$, resulta

$$\text{Var}\,(X) = 2\int_0^\infty x e^{-x^\beta}\, dx - [E(X)]^2.$$

4.

***b*)** Seja $f(x) = \begin{cases} xe^{-\frac{x^2}{2}} & \text{se } x > 0 \\ 0 & \text{se } x \leqslant 0. \end{cases}$

$$E(X) = \int_0^\infty x\, f(x)\, dx = \int_0^\infty x^2 e^{-\frac{x^2}{2}}\, dx = \int_0^{+\infty} \underbrace{x}_{f} \underbrace{(xe^{-\frac{x^2}{2}})}_{g'}\, dx.$$

Integrando por partes:

$$E(X) = \underbrace{\lim_{s\to\infty} \left[-xe^{-\frac{x^2}{2}} \right]_0^s}_{0} + \int_0^\infty e^{-\frac{x^2}{2}}\, dx$$

Fazendo a mudança de variável:

$$\frac{x}{\sqrt{2}} = u, \quad dx = \sqrt{2}\, du$$

$$E(X) = \int_0^\infty e^{-u^2} \sqrt{2}\, du = \sqrt{2} \int_0^\infty e^{-u^2}\, du = \sqrt{2} \cdot \frac{\sqrt{\pi}}{2} = \frac{\sqrt{2\pi}}{2}.$$

$\text{Var}\,(X) = \int_0^\infty x^2\, f(x)\, dx - [E(X)]^2$, ou seja,

$$\text{Var}\,(X) = \int_0^\infty x^3 e^{-\frac{x^2}{2}}\, dx - [E\,(X)]^2.$$

Integrando por partes:

$$\text{Var}\,(X) = \lim_{s\to\infty} \left\{ \left[-x^2 e^{-\frac{x^2}{2}} \right]_0^s + 2\int_0^s e^{-\frac{x^2}{2}} x\, dx \right\} - [E(X)]^2$$

$$\text{Var}\,(X) = \lim_{s\to\infty} \left\{ \left[-x^2 e^{-\frac{x^2}{2}} \right]_0^s - 2\left[e^{-\frac{x^2}{2}} \right]_0^s \right\} - \left(\frac{\sqrt{2\pi}}{2} \right)^2$$

Como $\lim_{s \to \infty} s^2 e^{-\frac{s^2}{2}} = 0$ e $\lim_{s \to \infty} e^{-\frac{s^2}{2}} = 0$, temos:

$$\text{Var}\,(X) = 2 - \frac{\pi}{2} = \frac{4 - \pi}{2}.$$