CAPÍTULO 7

Exercícios 7.3

1. Sejam $\vec{F}(t) = (t, \text{sen}\, t, 2)$ e $\vec{G}(t) = (3, t, t^2)$.

a) $\vec{F}(t) \cdot \vec{G}(t) = (t, \text{sen}\, t, 2) \cdot (3, t, t^2) = 3t + t\,\text{sen}\, t + 2t^2.$

d) $\vec{F}(t) \wedge \vec{G}(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ t & \text{sen}\, t & 2 \\ 3 & t & t^2 \end{vmatrix} = (t^2\,\text{sen}\, t - 2t)\,\vec{i} + (6 - t^3)\,\vec{j} + (t^2 - 3\,\text{sen}\, t)\,\vec{k}\,.$

$\vec{F}(t) \wedge \vec{G}(t) = (t^2\,\text{sen}\, t - 2t, 6 - t^3, t^2 - 3\,\text{sen}\, t).$

2. Sejam $\vec{r}(t) = t\vec{i} + 2\vec{j} + t^2\vec{k}$ e $\vec{x}(t) = t\vec{i} - \vec{j} + \vec{k}$.

$\vec{r}(t) \wedge \vec{x}(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ t & 2 & t^2 \\ t & -1 & 1 \end{vmatrix} = (2 + t^2)\,\vec{i} + (t^3 - t)\,\vec{j} + (-3t)\,\vec{k}\,.$

3. Sejam $\vec{u}(t) = \text{sen}\, t\,\vec{i} + \cos t\,\vec{j} + t\,\vec{k}$ e
$\vec{v}(t) = \text{sen}\, t\,\vec{i} + \cos t\,\vec{j} + \vec{k}$

$\vec{u}(t) \cdot \vec{v}(t) = \text{sen}^2\, t + \cos^2\, t + t = 1 + t\,.$

Exercícios 7.4

1. ***c)*** $\lim\limits_{t \to 2} \vec{r}(t) = \left(\lim\limits_{t \to 2} \dfrac{t^3 - 8}{t^2 - 4}\right)\vec{i} + \left(\lim\limits_{t \to 2} \dfrac{\cos \frac{\pi}{t}}{t - 2}\right)\vec{j} + \left(\lim\limits_{t \to 2} 2t\right)\vec{k}\,.$

$$\lim_{t \to 2} \frac{t^3 - 8}{t^2 - 4} = \lim_{t \to 2} \frac{(t - 2)(t^2 + 2t + 4)}{(t - 2)(t + 2)} = \frac{12}{4} = 3,$$

$$\lim_{t \to 2} \frac{\cos \frac{\pi}{t}}{t - 2} = \lim_{t \to 2} \frac{(-\text{sen}\, \frac{\pi}{t})(-\frac{\pi}{t^2})}{1} = \frac{\pi}{4} \cdot \text{sen}\, \frac{\pi}{2} = \frac{\pi}{4} \text{ e}$$

$$\lim_{t \to 2} 2t = 4\,.$$

Portanto,

$$\lim_{t\to 2} \vec{r}(t) = 3\vec{i} + \frac{\pi}{4}\vec{j} + 4\vec{k}\,.$$

2. *b*) $f(t)\cdot\vec{F}(t) = \underbrace{f(t)}_{\in\mathbb{R}}\cdot\underbrace{(F_1(t), F_2(t), ..., F_n(t))}_{\in\mathbb{R}^n}$. Temos

$f(t)\cdot\vec{F}(t) = (f(t)\,F_1(t), f(t)\,F_2(t), ..., f(t)\,F_n(t))$. Segue que

$$\lim_{t\to t_0} f(t)\,\vec{F}(t) = (\lim_{t\to t_0} f(t)\,F_1(t), \lim_{t\to t_0} f(t)\,F_2(t), ..., \lim_{t\to t_0} f(t)\,F_n(t))$$

$$= (\lim_{t\to t_0} f(t)\cdot\lim_{t\to t_0} F_1(t), \lim_{t\to t_0} f(t)\lim_{t\to t_0} F_2(t), ..., \lim_{t\to t_0} f(t)\cdot\lim_{t\to t_0} F_n(t)).$$

De $\lim_{t\to t_0} \vec{F}(t) = \vec{a}$ segue que $\lim_{t\to t_0} \| F(t) - \vec{a} \| = 0$.

Por outro lado, para todo $i = 1, 2, ..., n$

$| F_i(t) - a_i | \leqslant \| F(t) - a \|$.

Pelo Teorema do Confronto:

$\lim_{t\to t_0} F_i(t) = a_i$ para $i = 1, 2, ..., n$.

Portanto, usando $\lim_{t\to t_0} f(t) = L$ e $\lim_{t\to t_0} F_i(t) = a_i$, segue:

$$\lim_{t\to t_0} f(t)\,\vec{F}(t) = (La_1, La_2, ..., La_n) = L(a_1, a_2, ..., a_n) = L\vec{a}.$$

***c*)** Sejam $\vec{F}(t) = (F_1(t), F_2(t), F_3(t))$ e $\lim_{t\to t_0} \vec{F}(t) = \vec{a} = (a_1, a_2, a_3)$

$\vec{G}(t) = (G_1(t), G_2(t), G_3(t))$ e $\lim_{t\to t_0} \vec{G}(t) = \vec{b} = (b_1, b_2, b_3)$.

$$\vec{F}(t)\wedge\vec{G}(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ F_1(t) & F_2(t) & F_3(t) \\ G_1(t) & G_2(t) & G_3(t) \end{vmatrix}$$

$$\vec{F}(t)\wedge\vec{G}(t) = (F_2(t)\,G_3(t) - F_3(t)(G_2(t))\,\vec{i} + (F_3(t)\,G_1(t) - F_1(t)\,G_3(t))\,\vec{j}$$

$$+ (F_1(t)\,G_2(t) - F_2(t)\,G_2(t))\,\vec{k}.$$

De $\lim_{t\to t_0} \vec{F}(t) = \vec{a}$ segue que $\lim_{t\to t_0} F_i(t) = a_i$ para $i = 1, 2, 3$.

De $\lim_{t\to t_0} G(t) = \vec{b}$ segue que $\lim_{t\to t_0} G_i(t) = b_i$ para $i = 1, 2, 3$.

Temos

$$\lim_{t\to t_0} \vec{F}(t)\wedge\vec{G}(t)=\left[\lim_{t\to t_0}(F_2(t)\,G_3(t)-F_3(t)\,G_2(t)\right]\vec{i}$$

$$+\left[\lim_{t\to t_0}(F_3(t)\,G_1(t)-F_1(t)\,G_3(t))\right]\vec{j}+\left[\lim_{t\to t_0}(F_1(t)\,G_2(t)-F_2(t)\,G_1(t))\right]\vec{k}$$

$$=\left(\lim_{t\to t_0}F_2(t)\lim_{t\to t_0}G_3(t)-\lim_{t\to t_0}F_3(t)\lim_{t\to t_0}G_2(t)\right)\vec{i}+\left(\lim_{t\to t_0}F_3(t)\lim_{t\to t_0}G_1(t)\right.$$

$$\left.-\lim_{t\to t_0}F_1(t)\lim_{t\to t_0}G_3(t)\right)\vec{j}+\left(\lim_{t\to t_0}F_1(t)\lim_{t\to t_0}G_2(t)-\lim_{t\to t_0}F_2(t)\lim_{t\to t_0}G_1(t)\right)\vec{k}$$

$$=(a_2\,b_3-a_3\,b_2)\vec{i}+(a_3\,b_1-a_1\,b_3)\vec{j}+(a_1\,b_2-a_2\,b_1)\vec{k}$$

$$=\begin{vmatrix} a_2 & a_3\\ b_2 & b_3\end{vmatrix}\vec{i}+\begin{vmatrix} a_3 & a_1\\ b_3 & b_1\end{vmatrix}\vec{j}+\begin{vmatrix} a_1 & a_2\\ b_1 & b_2\end{vmatrix}\vec{k}$$

$$=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}=\vec{a}\wedge\vec{b}.$$

3. *b*) Seja $F(t) = \sqrt{t-1}\,\vec{i} + \sqrt{t+1}\,\vec{j} + e^t\,\vec{k}$
onde $F(t) = (F_1(t), F_2(t), F_3(t))$.

Componentes de F: $\begin{cases} F_1(t)=\sqrt{t-1}\\ F_2(t)=\sqrt{t+1}\\ F_3(t)=e^t.\end{cases}$

F é contínua em $t_0 \Leftrightarrow F_i$ é contínua em t_0 para $i = 1, 2, 3$.
$F_1(t)$ é contínua para $t - 1 \geqslant 0 \Rightarrow t \geqslant 1$,
$F_2(t)$ é contínua para $t + 1 \geqslant 0 \Rightarrow t \geqslant -1$ e
$F_3(t)$ é contínua para todo $t \in \mathbb{R}$.
Portanto,
F é contínua no conjunto $\{t \in \mathbb{R} \mid t \geqslant 1\}$.

Exercícios 7.5

1. *a*) $\vec{F}(t)=(3t^2, e^{-t}, \ln(t^2+1))$.

$$\frac{d\vec{F}}{dt}(t)=((3t^2)', (e^{-t})', (\ln(t^2+1))'),\text{ ou seja,}$$

$$\frac{d\vec{F}}{dt}(t)=\left(6t, -e^{-t}, \frac{2t}{t^2+1}\right).$$

$$\frac{d^2F}{dt^2}(t) = \left((6t)', (-e^{-t})', \left(\frac{2t}{t^2+1} \right)' \right), \text{ ou seja,}$$

$$\frac{d^2F}{dt^2}(t) = \left(6, e^{-t}, \frac{2-2t^2}{(t^2+1)^2} \right).$$

***c*)** $\vec{F}(t) = \text{sen } 5t\, \vec{i} + \cos 4t\, \vec{j} - e^{-2t}\, \vec{k}$.

$$\frac{d\vec{F}}{dt}(t) = \frac{d}{dt}(\text{sen } 5t)\, \vec{i} + \frac{d}{dt}(\cos 4t)\, \vec{j} - \frac{d}{dt}(e^{-2t})\, \vec{k}, \text{ ou seja,}$$

$$\frac{d\vec{F}}{dt}(t) = 5\cos 5t\, \vec{i} - 4 \text{ sen } 4t\, \vec{j} + 2e^{-2t}\, \vec{k}.$$

$$\frac{d^2F}{dt^2}(t) = \frac{d}{dt}(5\cos 5t)\, \vec{i} - \frac{d}{dt}(4 \text{ sen } 4t)\, \vec{j} + \frac{d}{dt}(2e^{-2t})\, \vec{k}, \text{ ou seja,}$$

$$\frac{d^2F}{dt^2}(t) = -\,25 \text{ sen } 5t\, \vec{i} - 16\cos 4t\, \vec{j} - 4\, e^{-2t}\, \vec{k}.$$

2. *b*) Sejam $G(t) = (t^2, t)$ e $G(1)$.

$\frac{dG}{dt}(t) = (2t, 1) \Rightarrow (2, 1)$ é o vetor tangente à trajetória de G, em $G(1)$.

Então, $X = G(1) + \lambda \frac{dG}{dt}(1)$ é a reta tangente à trajetória de G no ponto $G(1) = (1, 1)$.
Logo $(x, y) = (1, 1) + \lambda\, (2, 1)$; $\lambda \in \mathbb{R}$, é a reta procurada.

***c*)** Seja $F(t) = \left(\frac{1}{t}, \frac{1}{t}, t^2 \right)$.

$\frac{dF}{dt}(t) = \left(-\frac{1}{t^2}, -\frac{1}{t^2}, 2t \right)$, daí

$\frac{dF}{dt}(2) = \left(-\frac{1}{4}, -\frac{1}{4}, 4 \right)$ é o vetor tangente à trajetória de F no ponto $F(2)$.

Reta tangente:

$(x, y, z) = F(2) + \lambda \frac{dF}{dt}(2)$, ou seja,

$$(x, y, z) = \left(\frac{1}{2}, \frac{1}{2}, 4 \right) + \lambda \left(-\frac{1}{4}, -\frac{1}{4}, 4 \right).$$

4. Como $\vec{F} : I \to \mathbb{R}^3$ é derivável até 2.ª ordem em I, temos:

$$\frac{d}{dt}\left(\vec{F}(t) \wedge \frac{d\vec{F}}{dt}(t) \right) = \underbrace{\frac{d\vec{F}}{dt}(t) \wedge \frac{d\vec{F}}{dt}(t)}_{0} + \vec{F}(t) \wedge \frac{d^2F}{dt^2}(t).$$

Por hipótese: $\frac{d^2 F}{dt^2}(t) = \lambda \vec{F}(t)$.

Então $\frac{d}{dt}\left(\vec{F}(t) \wedge \frac{d\vec{F}}{dt}(t)\right) = \vec{F}(t) \wedge \lambda \vec{F}(t) = \vec{0}$, para todo t em I.

Logo, $\vec{F}(t) \wedge \frac{d\vec{F}}{dt}(t) = k$ (constante) em I.

6. $\vec{r}(t) \wedge \frac{d\vec{r}}{dt}(t) = k$. Daí

$$\underbrace{\frac{d}{dt}\left(\vec{r}(t) \wedge \frac{d\vec{r}}{dt}(t)\right)} = \frac{d}{dt}(k), \text{ ou seja,}$$

$$\vec{r}(t) \wedge \frac{d^2 r}{dt^2}(t) + \underbrace{\frac{d\vec{r}}{dt}(t) \wedge \frac{d\vec{r}}{dt}(t)}_{0} = \vec{0}$$

Logo $r(t) \wedge \frac{d^2 \vec{r}}{dt^2}(t) = \vec{0}$ em $\mathbb{R}$.

Exercícios 7.6

1. *b*) $\int_{-1}^{1}\left[\text{sen } 3t\,\vec{i} + \frac{1}{1+t^2}\,\vec{j} + \vec{k}\right] dt$

$$= \left[\int_{-1}^{1} \text{sen } 3t\, dt\right]\vec{i} + \left[\int_{-1}^{1} \frac{dt}{1+t^2}\right]\vec{j} + \left[\int_{-1}^{1} dt\right]\vec{k}$$

$$= \left[-\frac{1}{3}\cos 3t\right]_{-1}^{1}\vec{i} + \left[\text{arctg } t\right]_{-1}^{1}\vec{j} + [t]_{-1}^{1}\,\vec{k}$$

$$= \left[-\frac{1}{3}\cos 3 + \frac{1}{3}\underbrace{\cos(-3)}_{\cos 3}\right]\vec{i} + [\text{arctg } 1 - \text{arctg } (-1)]\,\vec{j}$$

$$+ [1-(-1)]\,\vec{k} = \left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right]\vec{j} + 2\vec{k} = \frac{\pi}{2}\,\vec{j} + 2\vec{k}.$$

2. *a*) $\vec{F}(t) \wedge \vec{G}(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ t & 1 & e^t \\ 1 & 1 & 1 \end{vmatrix} = (1-e^t)\vec{i} + (e^t - t)\vec{j} + (t-1)\,\vec{k}$.

$$\int_0^1 (\vec{F}(t) \wedge \vec{G}(t))\, dt = \left[\int_0^1 (1-e^t)\, dt\right]\vec{i} + \left[\int_0^1 (e^t - t)\, dt\right]\vec{j} + \left[\int_0^1 (t-1)\, dt\right]\vec{k}$$

$$= \left[t - e^t\right]_0^1 \vec{i} + \left[e^t - \frac{t^2}{2}\right]_0^1 \vec{j} + \left[\frac{t^2}{2} - t\right]_0^1 \vec{k}$$

$$= (2 - e)\vec{i} + \left(e - \frac{3}{2}\right)\vec{j} - \frac{1}{2}\vec{k}.$$

3. Seja $\vec{F} : [a, b] \to \mathbb{R}^n$ contínua,
$\vec{F}(t) = (F_1(t), F_2(t), ..., F_n(t))$.

Seja $\vec{G}(t) = \int_0^t \vec{F}(s)\, ds,\ t \in [a, b]$. Temos

$$\vec{G}(t) = \left(\underbrace{\int_0^t F_1(s)\, ds}_{G_1(t)}, \underbrace{\int_0^t F_2(s)\, ds}_{G_2(t)}, ..., \underbrace{\int_0^t F_n(s)\, ds}_{G_n(t)} \right)$$

Se $\vec{F} : [a, b] \to \mathbb{R}^n$ é contínua, então cada componente F_i de F é contínua.

Pelo Teorema Fundamental do Cálculo, sendo F_i definida e contínua no intervalo $[a, b]$, a função G_i dada por $G_i(t) = \int_0^t F_i(s)\, ds,\ t \in [a, b]$ $(i = 1, 2, ..., n)$ é uma primitiva de F_i em $[a, b]$, isto é, $G_i'(t) = F_i(t)$ para todo t em $[a, b]$.

Assim:

$$\frac{d\vec{G}}{dt}(t) = (\underbrace{G_1'(t)}_{F_1(t)}, \underbrace{G_2'(t)}_{F_2(t)}, ..., \underbrace{G_n'(t)}_{F_n(t)}) = F(t).$$

4. *a*) $I = \int_{t_1}^{t_2} \vec{F}(t)\, dt$. Temos

$$I = \int_0^2 (t\vec{i} + \vec{j} + t^2\vec{k})\, dt = \left[\int_0^2 t\, dt\right]\vec{i} + \left[\int_0^2 dt\right]\vec{j} + \left[\int_0^2 t^2\, dt\right]\vec{k}$$

$$= \left[\frac{t^2}{2}\right]_0^2 \vec{i} + [t]_0^2\, \vec{j} + \left[\frac{t^3}{3}\right]_0^2 \vec{k} = 2\vec{i} + 2\vec{j} + \frac{8}{3}\vec{k}.$$

Exercícios 7.7

1. *a*) $\gamma(t) = (t \cos t,\ t \operatorname{sen} t) \quad t \in [0, 2\pi]$. Daí

$\gamma'(t) = (-t \operatorname{sen} t + \cos t,\ t \cos t + \operatorname{sen} t)$ e, portanto,

$$\|\gamma'(t)\| = \sqrt{(-t \operatorname{sen} t + \cos t)^2 + (t \cos t + \operatorname{sen} t)^2}$$

$$= \sqrt{t^2(\operatorname{sen}^2 t + \cos^2 t) + (\operatorname{sen}^2 t + \cos^2 t)} = \sqrt{t^2 + 1}.$$

O comprimento da curva é:

$$L(\gamma) = \int_0^{2\pi} \|\gamma'(t)\|\, dt = \int_0^{2\pi} \sqrt{t^2+1}\; dt.$$

Façamos a mudança de variável:

$t = \text{tg}\, u;\;\; dt = \sec^2 u\, du$
$t = 0;\;\; u = 0$
$t = 2\pi;\;\; u = \text{arctg}\, 2\pi$

$$L(\gamma) = \int_0^{\text{arctg}\, 2\pi} \underbrace{\sqrt{1+\text{tg}^2 u}}_{\sec u}\, \sec^2 u\, du = \int_0^{\text{arctg}\, 2\pi} \sec^3 u\, du$$

$$= \int_0^{\text{arctg}\, 2\pi} \underset{\substack{\uparrow \\ f}}{\sec u} \cdot \underset{\substack{\uparrow \\ g'}}{\sec^2 u}\, du = \Big[\sec u\, \text{tg}\, u\Big]_0^{\text{arctg}\, 2\pi} - \int_0^{\text{arctg}\, 2\pi} \underbrace{\sec u\, \text{tg}\, u}_{f'}\, \underset{g}{\text{tg}\, u}\, du$$

$$= \Big[\sec u\, \text{tg}\, u\Big]_0^{\text{arctg}\, 2\pi} - \int_0^{\text{arctg}\, 2\pi} \sec u\, (\sec^2 u - 1)\, du$$

ou seja $2\displaystyle\int_0^{\text{arctg}\, 2\pi} \sec^3 u\, du = \Big[\sec u\, \text{tg}\, u\Big]_0^{\text{arctg}\, 2\pi} + \int_0^{\text{arctg}\, 2\pi} \sec u\, du$

Portanto:

$$L(\gamma) = \int_0^{\text{arctg}\, 2\pi} \sec^3 u\, du = \frac{1}{2}\Big[\sec u\, \text{tg}\, u\Big]_0^{\text{arctg}\, 2\pi} + \frac{1}{2}\Big[\ln(\sec u + \text{tg}\, u)\Big]_0^{\text{arctg}\, 2\pi}$$

$$= \frac{1}{2}\Big[\underbrace{\sec(\text{arctg}\, 2\pi)}_{\sqrt{1+4\pi^2}} \cdot 2\pi - \overset{1}{\cancel{\sec 0}}\; \overset{0}{\cancel{\text{tg}\, 0}}\Big] + \frac{1}{2}\Big[\ln\big(\underbrace{\sec(\text{arctg}\, 2\pi)}_{\sqrt{1+4\pi^2}} + 2\pi\big) - \underbrace{\ln(\overset{1}{\cancel{\sec 0}} + \overset{0}{\cancel{\text{tg}\, 0}})}_{0}\Big]$$

$$= \frac{1}{2}\sqrt{1+4\pi^2} \cdot 2\pi + \frac{1}{2}(\ln(\sqrt{1+4\pi^2} + 2\pi)), \text{ ou seja,}$$

$$L(\gamma) = \pi\sqrt{1+4\pi^2} + \frac{1}{2}\ln(2\pi + \sqrt{1+4\pi^2}).$$

c) $\gamma(t) = (\cos t, \text{sen}\, t, e^{-t})\; t \in [0, \pi]$.

$\gamma'(t) = (-\text{sen}\, t, \cos t, -e^{-t})$.

$\|\gamma'(t)\|^2 = \text{sen}^2\, t + \cos^2 t + e^{-2t} = 1 + e^{-2t}$.

$$L(\gamma) = \int_0^{\pi} \|\gamma'(t)\|\, dt = \int_0^{\pi} (1+e^{-2t})^{\frac{1}{2}}\, dt.$$

Façamos a mudança de variável

$e^{-t} = \text{tg}\ \theta;\quad -e^{-t}\,dt = \sec^2\theta\,d\theta;\quad dt = -\dfrac{\sec^2\theta}{\text{tg}\ \theta}\,d\theta$

$t = 0;\ \theta = \dfrac{\pi}{4}\ 1 + \text{tg}^2\,\theta = \sec^2\theta$

$t = \pi;\ \theta = \text{arctg}\ e^{-\pi}$.

Temos:

$$\int_0^{\pi}\sqrt{1+e^{-2t}}\,dt = \int_{\frac{\pi}{4}}^{\text{arctg}\ e^{-\pi}} -\frac{\sec^3\theta}{\text{tg}\ \theta}\,d\theta = \int_{\text{arctg}\ e^{-\pi}}^{\pi/4} \text{cossec}\ \theta\ \overbrace{\sec^2\theta}^{(1+\text{tg}^2\,\theta)}\ d\theta$$

$$= \int_{\text{arctg}\ e^{-\pi}}^{\pi/4} \text{cossec}\ \theta\,d\theta + \int_{\text{arctg}\ e^{-\pi}}^{\pi/4} \text{cossec}\ \theta\ \text{tg}^2\,\theta\,d\theta$$

$$= \int_{\text{arctg}\ e^{-\pi}}^{\pi/4} \text{cossec}\ \theta\,d\theta + \int_{\text{arctg}\ e^{-\pi}}^{\pi/4} (\cos\theta)^{-2}\ \text{sen}\ \theta\,d\theta$$

$$= \Big[\ln(\text{cossec}\ \theta - \cot\theta)\Big]_{\text{arctg}\ e^{-\pi}}^{\pi/4} + \left[-\frac{(\cos\theta)^{-1}}{(-1)}\right]_{\text{arctg}\ e^{-\pi}}^{\pi/4} \quad ①$$

Agora:

$$\text{cossec}\ \theta - \cot\theta = \frac{1}{\text{sen}\ \theta} - \frac{\cos\theta}{\text{sen}\ \theta} = \frac{1-\cos\theta}{\text{sen}\ \theta}$$

$$\frac{1-\cos{}^{\pi}\!/_{4}}{\text{sen}\ {}^{\pi}\!/_{4}} = \frac{1-\dfrac{\sqrt{2}}{2}}{\dfrac{\sqrt{2}}{2}} = \sqrt{2}-1 \quad ②$$

$$\frac{1-\cos(\text{arctg}\ e^{-\pi})}{\text{sen}(\text{arctg}\ e^{-\pi})} = \frac{1-\dfrac{1}{\sqrt{1+e^{-2\pi}}}}{\dfrac{e^{-\pi}}{\sqrt{1+e^{-2\pi}}}} = \frac{\sqrt{1+e^{-2\pi}}-1}{e^{-\pi}}\,\frac{(\sqrt{1+e^{-2\pi}}+1)}{(\sqrt{1+e^{-2\pi}}+1)}$$

$$= \frac{1+e^{-2\pi}-1}{e^{-\pi}(\sqrt{1+e^{-2\pi}}+1)} = \frac{e^{-\pi}}{\sqrt{1+e^{-2\pi}}+1} \quad ③$$

$$\Big[(\cos\theta)^{-1}\Big]_{\text{arctg}\ e^{-\pi}}^{\pi/4} = \left[\left(\cos\frac{\pi}{4}\right)^{-1} - (\cos \text{arctg}\ e^{-\pi})^{-1}\right]$$

$$= \left[\sqrt{2} - \left(\frac{1}{\sqrt{1+e^{-2\pi}}}\right)^{-1}\right] = \sqrt{2} - \sqrt{1+e^{-2\pi}} \quad ④$$

Substituindo ②, ③ e ④ em ①:

$$L(\gamma) = \ln(\sqrt{2}-1) - \ln\frac{e^{-2\pi}}{\sqrt{1+e^{-2\pi}}+1} + \sqrt{2} - \sqrt{1+e^{-2\pi}},$$

$$L(\gamma) = \ln(\sqrt{2}-1) - (-\pi)\underbrace{\ln e}_{1} + \ln(\sqrt{1+e^{-2\pi}}+1) + \sqrt{2} - \sqrt{1+e^{-2\pi}},$$

$$L(\gamma) = \ln(\sqrt{2}-1)\cdot(\sqrt{1+e^{-2\pi}}+1) + \pi + \sqrt{2} - \sqrt{1+e^{-2\pi}}$$

ou seja

$$L(\gamma) = \ln\frac{\sqrt{1+e^{-2\pi}}+1}{\sqrt{2}+1} + \pi + \sqrt{2} - \sqrt{1+e^{-2\pi}}.$$

f) Seja $\gamma:[0, \pi] \to \mathbb{R}^2$ tal que $x(t) = 1 - \cos t$ e $y(t) = t - \operatorname{sen} t$

$x'(t) = \operatorname{sen} t$ e $y'(t) = 1 - \cos t$

$\gamma'(t) = (\operatorname{sen} t, 1 - \cos t)$

$\|\gamma'(t)\|^2 = \operatorname{sen}^2 t + (1 - \cos t)^2 = 2(1 - \cos t) = 2 \cdot 2 \operatorname{sen}^2 \frac{t}{2} = 4 \operatorname{sen}^2 \frac{t}{2}$

$$L(\gamma) = \int_0^\pi \|\gamma'(t)\|\, dt = \int_0^\pi 2 \operatorname{sen}\frac{t}{2}\, dt = 4\left[-\cos\frac{t}{2}\right]_0^\pi$$

$$= 4\left(-\cos\frac{\pi}{2} + \cos 0\right) = 4.$$

6. ***a***) Seja $\gamma: [a, b] \to \mathbb{R}^n$, com derivada contínua e tal que $\|\gamma'(t)\| \neq 0$ em $[a, b]$.

Seja $s: [a, b] \to \mathbb{R}$ dada por $s(t) = \int_a^t \|\gamma'(u)\|\, du.$

Nestas condições a função $s = s(t)$ é inversível. Seja $t = t(s)$ sua inversa.

A curva $\delta: [0, L] \to \mathbb{R}^n$ dada por $\delta(s) = \gamma(t(s))$ está parametrizada pelo comprimento de arco (reparametrização de γ pelo comprimento de arco).

Portanto, $\gamma(t) = (2t + 1, 3t - 1)$ $t \geqslant 0$. Temos

$\gamma'(t) = (2, 3)$ e daí

$\|\gamma'(t)\| = \sqrt{4+9} = \sqrt{13}$. Segue

$$s(t) = \int_0^t \sqrt{13}\, du = \sqrt{13}\, t \Rightarrow t(s) = \frac{1}{\sqrt{13}} s. \text{ Daí}$$

$$\delta(s) = \gamma(t(s)) = \left(\frac{2s}{\sqrt{13}} + 1, \frac{3s}{\sqrt{13}} - 1\right).$$

b) $\gamma(t) = (2\cos t, 2 \operatorname{sen} t)$, $t \geqslant 0$.

$\gamma'(t) = (-2 \operatorname{sen} t, 2\cos t)$ e daí

$\|\gamma'(t)\| = \sqrt{4\operatorname{sen}^2 t + 4\cos^2 t} = 2$. Segue que

$$s(t) = \int_0^t 2\, du = 2t \Rightarrow t(s) = \frac{1}{2}s \text{ e, portanto,}$$

$$\delta(t) = \gamma(t(s)) = \left(2\cos\frac{s}{2}, 2\operatorname{sen}\frac{s}{2}\right).$$

d) $\gamma(t) = (e^t \cos t, e^t \operatorname{sen} t), \qquad t \geqslant 0$. Temos

$\gamma'(t) = (e^t \cos t - e^t \operatorname{sen} t, e^t \operatorname{sen} t + e^t \cos t)$ e daí

$\| \gamma'(t) \|^2 = e^{2t} [(\cos t - \operatorname{sen} t)^2 + (\operatorname{sen} t + \cos t)^2] = 2e^{2t}$, ou seja,

$\| \gamma'(t) \| = e^t \sqrt{2}$. Então

$$s(t) = \int_0^t e^u \sqrt{2}\, du = \sqrt{2}\, e^t - \sqrt{2} \quad \Rightarrow \quad e^t = \frac{s}{\sqrt{2}} + 1 \Rightarrow t(s) = \ln \frac{s + \sqrt{2}}{\sqrt{2}}, \text{ ou seja,}$$

$$\delta(s) = \gamma(t(s)) = \frac{s + \sqrt{2}}{\sqrt{2}} \left(\cos\left(\ln \frac{s + \sqrt{2}}{\sqrt{2}} \right), \operatorname{sen}\left(\ln \frac{s + \sqrt{2}}{\sqrt{2}} \right) \right).$$