CAPÍTULO 2

Exercícios 2.1

1. c. $s_n = \sum_{k=0}^{n} e^{-k} = 1 + \frac{1}{e} + \frac{1}{e^2} + ... + \frac{1}{e^n} = \frac{1 - \left(\frac{1}{e}\right)^{n+1}}{1 - \frac{1}{e}}.$

$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{1 - \left(\frac{1}{e}\right)^{n+1}}{1 - \frac{1}{e}} = \frac{e}{e-1}$, ou seja, $\sum_{k=0}^{\infty} e^{-k} = \frac{e}{e-1}.$

e) $s_n = \sum_{k=0}^{n} \frac{1}{(4k+1)(4k+5)} = \sum_{k=0}^{n} \frac{1}{4}\left(\underbrace{\frac{1}{4k+1}}_{b_k} - \underbrace{\frac{1}{4k+5}}_{b_{k+1}}\right) =$

$$= \frac{1}{4}\left[\left(1 - \frac{1}{5}\right) + \left(\frac{1}{5} - \frac{1}{9}\right) + \left(\frac{1}{9} - \frac{1}{13}\right) + ... + \left(\frac{1}{4n+1} - \frac{1}{4n+5}\right)\right] =$$

$$= \frac{1}{4}\left[1 - \frac{1}{4n+5}\right].$$

Como $\lim_{n\to\infty} \frac{1}{4n+5} = 0$, segue $\lim_{n\to\infty} s_n = \frac{1}{4}$, ou seja, $\sum_{k=0}^{\infty} \frac{1}{(4k+1)(4k+5)} = \frac{1}{4}.$

f) $s_n = \sum_{k=1}^{n} \frac{1}{k(k+1)(k+2)(k+3)} = \sum_{k=1}^{n} \frac{1}{3}\left[\frac{1}{k(k+1)(k+2)} - \frac{1}{(k+1)(k+2)(k+3)}\right] =$

$$= \frac{1}{3}\sum_{k=1}^{n}\left[\underbrace{\frac{1}{k(k+1)(k+2)}}_{b_k} - \underbrace{\frac{1}{(k+1)(k+2)(k+3)}}_{b_{k+1}}\right] =$$

$$= \frac{1}{3}\left[\left(\frac{1}{6} - \frac{1}{24}\right) + \left(\frac{1}{24} - \frac{1}{60}\right) + ... + \left(\frac{1}{n(n+1)(n+2)} - \frac{1}{(n+1)(n+2)(n+3)}\right)\right] =$$

$$= \frac{1}{3}\left[\frac{1}{6} - \frac{1}{(n+1)(n+2)(n+3)}\right].$$

Como $\lim\limits_{n\to\infty} \dfrac{1}{(n+1)(n+2)(n+3)} = 0$, segue que $\lim\limits_{n\to\infty} s_n = \dfrac{1}{3}\cdot\dfrac{1}{6} = \dfrac{1}{18}$, ou seja,

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)(k+2)(k+3)} = \frac{1}{18}.$$

g) $s_n = \displaystyle\sum_{k=1}^{n} \frac{2k+1}{k^2(k+1)^2} = \sum_{k=1}^{n} \left[\underbrace{\frac{1}{k^2}}_{b_k} - \underbrace{\frac{1}{(k+1)^2}}_{b_{k+1}} \right] = \left[1 - \frac{1}{(n+1)^2}\right]$

Logo $\lim\limits_{n\to\infty} s_n = 1$, ou seja, $\displaystyle\sum_{k=1}^{\infty} \frac{2k+1}{k^2(k+1)^2} = 1.$

h) $\displaystyle\sum_{n=1}^{\infty} n\alpha^n = \alpha + 2\alpha^2 + 3\alpha^3 + 4\alpha^4 + ... =$

$$= \underbrace{(\alpha + \alpha^2 + \alpha^3 + \alpha^4 + ...)} + \underbrace{(\alpha^2 + \alpha^3 + \alpha^4 + ...)} + \underbrace{(\alpha^3 + \alpha^4 + ...)} + ... =$$

$$= \frac{\alpha}{1-\alpha} + \frac{\alpha^2}{1-\alpha} + \frac{\alpha^3}{1-\alpha} + ... =$$

$$= \frac{1}{1-\alpha} \underbrace{(\alpha + \alpha^2 + \alpha^3 + ...)}_{\frac{\alpha}{1-\alpha}} = \frac{\alpha}{(1-\alpha)^2}.$$

i) $s_n = \displaystyle\sum_{k=1}^{n} \frac{1}{k(k+1)(k+2)...(k+p)}$, onde $p \geqslant 1$ natural.

$$s_n = \sum_{k=1}^{n} \frac{1}{p} \left[\underbrace{\frac{1}{k(k+1)(k+2)...(k+p-1)}}_{b_k} - \underbrace{\frac{1}{(k+1)(k+2)...(k+p)}}_{b_{k+1}} \right]$$

(série telescópica)

$$s_n = \frac{1}{p} \left[\underbrace{\frac{1}{1\cdot 2\cdot 3\cdot ...\cdot p}}_{b_1} - \underbrace{\frac{1}{(n+1)(n+2)...(n+p)}}_{b_n} \right].$$

Como $\lim_{n\to\infty} b_n = 0$, segue que

$$\lim_{n\to\infty} s_n = \frac{1}{p}\left[\frac{1}{1\cdot 2\cdot 3\cdot \ldots p}\right] = \frac{1}{p}\cdot\frac{1}{p!} = \frac{1}{p\cdot p!},$$

ou seja, $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k(k+1)(k+2)\ldots(k+p)} = \frac{1}{p\cdot p!}$.

j) $$\sum_{k=0}^{\infty} \frac{1}{(4k+1)(4k+3)} = \frac{1}{2}\sum_{k=0}^{\infty}\left[\frac{1}{4k+1} - \frac{1}{4k+3}\right] =$$

$$= \frac{1}{2}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\right) = \frac{1}{2}\underbrace{\sum_{k=0}^{\infty}(-1)^k\frac{1}{2k+1}}_{\text{arc tg } 1} = \frac{1}{2}\cdot\frac{\pi}{4} = \frac{\pi}{8}.$$

l) $$s_n = \sum_{k=1}^{n} \frac{1}{k^2(k+1)(k+2)^2} = \sum_{k=1}^{n}\frac{1}{4}\left[\underbrace{\frac{1}{k^2(k+1)^2}}_{b_k} - \underbrace{\frac{1}{(k+1)^2(k+2)^2}}_{b_{k+1}}\right] =$$

(série telescópica)

$$= \frac{1}{4}\left[\underbrace{\frac{1}{4}}_{b_1} - \underbrace{\frac{1}{(n+1)^2(n+2)^2}}_{b_{n+1}}\right].$$

Como $\displaystyle\lim_{n\to\infty} \frac{1}{(n+1)^2(n+2)^2} = 0$ segue que

$\displaystyle\lim_{n\to\infty} s_n = \frac{1}{4}\cdot\frac{1}{4} = \frac{1}{16}$, ou seja,

$$\sum_{k=1}^{\infty} \frac{1}{k^2(k+1)(k+2)^2} = \frac{1}{16}.$$

2. ***a***) Vamos mostrar que $\displaystyle\ln(1+\alpha) = \sum_{k=1}^{\infty} (-1)^{k+1}\frac{\alpha^k}{k}$.

Pela progressão geométrica:

$$1 + r + r^2 + ... + r^{n-1} = \frac{1 - r^n}{1 - r}$$

$$\frac{1}{1-r} = 1 + r + r^2 + ... + r^{n-1} + \frac{r^n}{1-r}.$$

Fazendo $r = -x$,

$$\frac{1}{1+x} = 1 - x + x^2 + ... + (-1)^{n-1} x^{n-1} + (-1)^n \frac{x^n}{x+1}.$$

Como $\ln(1+\alpha) = \int_0^\alpha \frac{1}{1+x}\, dx$ resulta, integrando termo a termo:

$$\ln(1+\alpha) = \alpha - \frac{\alpha^2}{2} + \frac{\alpha^3}{3} + ... + (-1)^{n-1} \frac{\alpha^n}{n} + (-1)^n \int_0^\alpha \frac{x^n}{1+x}\, dx.$$

Ou seja, $\ln(1+\alpha) = \sum_{k=1}^{n} (-1)^{k-1} \frac{\alpha^k}{k} + (-1)^n \int_0^\alpha \frac{x^n}{1+x}\, dx.$ ①

Temos que mostrar que, para $0 < \alpha \leqslant 1$, segue

$$\lim_{n\to\infty} \int_0^\alpha \frac{x^n}{1+x}\, dx = 0.$$

Temos $0 \leqslant \frac{x^n}{1+x} \leqslant x^n$ para $x \in [0, \alpha]$.
Daí,

$$0 \leqslant \int_0^\alpha \frac{x^n}{1+x}\, dx \leqslant \int_0^\alpha x^n\, dx$$

$$0 \leqslant \int_0^\alpha \frac{x^n}{1+x}\, dx \leqslant \frac{\alpha^{n+1}}{n+1}. \quad ②$$

De $0 < \alpha \leqslant 1$ segue que $\lim_{n\to\infty} \frac{\alpha^{n+1}}{n+1} = 0.$

Portanto, $\lim_{n\to\infty} \int_0^\alpha \frac{x^n}{1+x}\, dx = 0.$

Substituindo em ①, concluímos que:

$$\ln(1+\alpha) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\alpha^k}{k}. \quad ③$$

b) De ①, resulta

$$\left|\ln(1+\alpha) - \sum_{k=1}^{n} (-1)^{k-1}\,\frac{\alpha^k}{k}\right| = \left|(-1)^n \int_0^{\alpha} \frac{x^n}{1+x}\,dx\right|$$

Tendo em vista ②,

$$\left|\ln(1+\alpha) - \sum_{k=1}^{n} (-1)^{k-1}\,\frac{\alpha^k}{k}\right| \leqslant \frac{\alpha^{n+1}}{n+1}.$$

3. ***b***) $\frac{1}{2} - \frac{1}{2\cdot 2^2} + \frac{1}{3\cdot 2^3} - \frac{1}{4\cdot 2^4} + \ldots = \sum_{k=1}^{\infty} (-1)^{k+1}\,\frac{1}{k\cdot 2^k} =$

$$= \sum_{k=1}^{\infty} (-1)^{k+1}\,\frac{(2^{-1})^k}{k} = \ln\left(1+\frac{1}{2}\right) = \ln\frac{3}{2}.$$

c) $\frac{1}{3} - \frac{1}{2\cdot 3^2} + \frac{1}{3\cdot 3^3} = \frac{1}{4\cdot 3^4} + \ldots = \sum_{k=1}^{\infty} (-1)^{k+1}\,\frac{1}{k\cdot 3^k} =$

$$= \sum_{k=1}^{\infty} (-1)^{k+1}\,\frac{(3^{-1})^k}{k} = \ln\left(1+\frac{1}{3}\right) = \ln\frac{4}{3}.$$

4. Utilizando o Exercício 2,

$$\ln\frac{6}{5} = \ln\left(1+\frac{1}{5}\right) = \sum_{k=1}^{\infty} (-1)^{k+1}\,\frac{1}{k\cdot 5^k}$$

$$\ln\frac{5}{4} = \ln\left(1+\frac{1}{4}\right) = \sum_{k=1}^{\infty} (-1)^{k+1}\,\frac{1}{k\cdot 4^k}$$

$$\ln\frac{4}{3} = \ln\left(1+\frac{1}{3}\right) = \sum_{k=1}^{\infty} (-1)^{k+1}\,\frac{1}{k\cdot 3^k}$$

Utilizando a relação $\ln 2 = \ln\frac{6}{5} + \ln\frac{5}{4} + \ln\frac{4}{3}$, segue:

$$\ln 2 = \sum_{k=1}^{\infty} (-1)^{k+1}\,\frac{1}{k}\left(\frac{1}{5^k} + \frac{1}{4^k} + \frac{1}{3^k}\right).$$

***b*)** Temos

$$\ln 5 = \ln 4 \cdot \frac{5}{4} = 2 \ln 2 + \ln \frac{5}{4} = 2\left\{\ln \frac{6}{5} + \ln \frac{5}{4} + \ln \frac{4}{3}\right\} + \ln \frac{5}{4}$$

ou seja, $\ln 5 = 2 \ln \frac{6}{5} + 3 \ln \frac{5}{4} + 2 \ln \frac{4}{3}$.

Daí,

$$\ln 5 = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2}{k \cdot 5^k} + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{3}{k \cdot 4^k} + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{2}{k \cdot 3^k}$$

ou seja,

$$\ln 5 = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k}\left(\frac{2}{5^k} + \frac{3}{4^k} + \frac{2}{3^k}\right).$$

***c*)** Temos

$$\ln 7 = \ln\left(6 \cdot \frac{7}{6}\right) = \ln 6 + \ln \frac{7}{6} = \ln 6 + \ln\left(1 + \frac{1}{6}\right) =$$

$$= \ln 6 + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k \cdot 6^k}.$$

***d*)** Temos

$$\ln (p+1) = \ln\left(p \cdot \frac{p+1}{p}\right) = \ln p + \ln\left(1 + \frac{1}{p}\right) =$$

$$= \ln p + \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{kp^k} \quad (p \geqslant 1).$$

7. *b*) $\ln 3 = \ln 4 + \ln \frac{3}{4} = \ln 4 + \ln\left(1 - \frac{1}{4}\right) = \ln 4 - \sum_{k=1}^{\infty} \frac{1}{k \cdot 4^k}.$

8. Por 7a, temos $\ln 2 = -\ln\left(1 - \frac{1}{2}\right) = \sum_{k=1}^{\infty} \frac{1}{k \cdot 2^k}.$

Por 6b, temos $\left|\ln (1-\alpha) + \sum_{k=1}^{n} \frac{\alpha^k}{k}\right| \leqslant \frac{\alpha^{n+1}}{(n+1)(1-\alpha)}.$

Daí, para $\alpha = \dfrac{1}{2}$, segue:

$$\left|\ln\left(1-\frac{1}{2}\right)+\sum_{k=1}^{n}\frac{1}{k\cdot 2^k}\right| \leqslant \frac{\left(\frac{1}{2}\right)^{n+1}}{(n+1)\cdot\left(\frac{1}{2}\right)} = \frac{1}{2^n\,(n+1)}.$$

Como o erro, em módulo, deve ser inferior a 10^{-5},

$\dfrac{1}{2^n(n+1)} < 10^{-5}$, ou seja, $2^n\,(n+1) > 10^5$.

Por tentativas, basta tomar $n \geqslant 13$.

9. ***a***) Pelo Exemplo 7, temos

$1 + r + r^2 + ... + r^n = \dfrac{1-r^{n+1}}{1-r}$ e daí

$$\frac{1}{1-r} = 1 + r + r^2 + ... + r^n + \frac{r^{n+1}}{1-r}.$$

Fazendo $r = x^2$,

$$\frac{1}{1-x^2} = 1 + x^2 + x^4 + ... + x^{2n} + \frac{x^{2n+2}}{1-x^2}.$$

Como $\ln\left(\dfrac{1+\alpha}{1-\alpha}\right) = 2\displaystyle\int_0^\alpha \frac{1}{1-x^2}\,dx$, resulta (integrando)

$$\ln\frac{1+\alpha}{1-\alpha} = 2\left(\alpha + \frac{\alpha^3}{3} + \frac{\alpha^5}{5} + ... + \frac{\alpha^{2n+1}}{2n+1}\right) + 2\int_0^\alpha \frac{x^{2n+2}}{1-x^2}\,dx \quad ①$$

onde $\left|\displaystyle\int_0^\alpha \frac{x^{2n+2}}{1-x^2}\,dx\right| \leqslant \dfrac{\alpha^{2n+3}}{2n+3}\cdot\dfrac{1}{1-\alpha^2}$ ②, $0 < |\alpha| < 1$.

Portanto,

$$\lim_{n\to\infty}\int_0^\alpha \frac{x^{2n+2}}{1-x^2}\,dx = 0.$$

Então,

$$\ln\frac{1+\alpha}{1-\alpha} = 2\sum_{k=0}^{\infty}\frac{\alpha^{2k+1}}{2k+1}.$$

b) De ① e ②, temos

$$\left|\ln\frac{1+\alpha}{1-\alpha}-2\sum_{k=0}^{n}\frac{\alpha^{2k+1}}{2k+1}\right|=\left|2\int_0^{\alpha}\frac{x^{2n+2}}{1-x^2}\,dx\right|\leqslant\frac{2}{2n+3}\cdot\frac{\alpha^{2n+3}}{1-\alpha^2}.$$

10. ***b***) Pelo Exercício 9a, temos:

$\ln\frac{1+\alpha}{1-\alpha}=2\sum_{k=0}^{\infty}\frac{\alpha^{2k+1}}{2k+1}$. Como $\beta=\frac{1+\alpha}{1-\alpha}$, segue $\ln\beta=2\sum_{k=0}^{\infty}\frac{\alpha^{2k+1}}{2k+1}$.

11. ***a***) Temos $\alpha=\frac{\beta-1}{\beta+1}=\frac{1}{3}$. Portanto,

$$\ln 2=2\sum_{k=0}^{\infty}\frac{\alpha^{2k+1}}{2k+1}=2\sum_{k=0}^{\infty}\frac{1}{(2k+1)}\cdot\frac{1}{3^{2k+1}}.$$

b) $\ln(p+1)=\ln\left(p\cdot\frac{p+1}{p}\right)=\ln p+\ln\underbrace{\left(1+\frac{1}{p}\right)}_{\beta}.$

Temos $\alpha=\frac{\beta-1}{\beta+1}=\frac{1}{2p+1}$.

Portanto,

$$\ln(p+1)=\ln p+2\sum_{k=0}^{\infty}\frac{1}{(2k+1)(2p+1)^{2k+1}}.$$

12. Pelo Exercício 11a, temos: $\ln 2=2\sum_{k=0}^{\infty}\frac{1}{(2k+1)\,3^{2k+1}}$.

Pelo Exercício 9b, temos

$$\left|\ln 2-2\sum_{k=0}^{n}\frac{1}{(2k+1)\cdot 3^{2k+1}}\right|\leqslant\frac{2}{2n+3}\cdot\frac{\alpha^{2n+3}}{1-\alpha^2}.$$

Portanto,

$$\frac{2}{2n+3}\cdot\left(\frac{1}{3}\right)^{2n+3}\cdot\frac{9}{8}\leqslant 10^{-5}$$

basta tomar $n\geqslant 4$.

Comparando com o Exercício 8 ($n \geqslant 13$), verificamos que, nesse caso, a convergência é muito mais rápida.

13. ***a***) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{2k(2k-1)} = \sum_{k=1}^{\infty}\left[\frac{1}{2k-1} - \frac{1}{2k}\right] = \left(1-\frac{1}{2}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{5}-\frac{1}{6}\right)+\dots$

Sabemos que

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots = \sum_{k=1}^{\infty} (-1)^{k+1}\frac{1}{k}.$$

Daí,

$$\ln 2 = \lim_{n\to\infty} \sum_{k=1}^{2n} (-1)^{k+1}\frac{1}{k} = \sum_{k=1}^{\infty}\frac{1}{2k(2k-1)}.$$

b) $\displaystyle\sum_{k=0}^{\infty}\frac{1}{(4k+1)(4k+3)} = \sum_{k=0}^{\infty}\frac{1}{2}\left[\frac{1}{4k+1} - \frac{1}{4k+3}\right] = \frac{1}{2}\left[\left(1-\frac{1}{3}\right)+\left(\frac{1}{5}-\frac{1}{7}\right)+\dots\right]$

Sabemos, pelo Exemplo 8, que

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots.$$

Então,

$$\sum_{k=0}^{\infty}\frac{1}{(4k+1)(4k+3)} = \frac{1}{2}\left[\left(1-\frac{1}{3}\right)+\left(\frac{1}{5}-\frac{1}{7}\right)+\dots\right] = \frac{\pi}{8}.$$

16. Temos

$\text{arc tg}\,\dfrac{1}{5} = 2\ \text{arc tg}\,\dfrac{1}{10} + \text{arc tg}\,\beta$ ①

Daí,

$\text{arc tg}\,\dfrac{1}{5} - \text{arc tg}\,\dfrac{1}{10} = \text{arc tg}\,\dfrac{1}{10} + \text{arc tg}\,\beta.$

Pela fórmula $\text{tg}\,(a+b) = \dfrac{\text{tg}\,a + \text{tg}\,b}{1 - \text{tg}\,a\,\text{tg}\,b}$ segue

$$\frac{\frac{1}{5}-\frac{1}{10}}{1+\frac{1}{5}\cdot\frac{1}{10}} = \frac{\frac{1}{10}+\beta}{1-\frac{\beta}{10}} \Rightarrow \frac{5}{51} = \frac{1+10\beta}{10-\beta} \Rightarrow \beta = -\frac{1}{515}.$$

Substituindo em ①,

$$\text{arc tg } \frac{1}{5} = 2 \text{ arc tg } \frac{1}{10} - \text{arc tg } \frac{1}{515} \quad ②$$

Pela fórmula de Machin, $\frac{\pi}{4} = 4 \text{ arc tg } \frac{1}{5} - \text{arc tg } \frac{1}{239}$ ③

Substituindo ② em ③ segue

$$\frac{\pi}{4} = 4\left(2 \text{ arc tg } \frac{1}{10} - \text{arc tg } \frac{1}{515}\right) - \text{arc tg } \frac{1}{239}$$

ou seja, $\frac{\pi}{4} = 8 \text{ arc tg } \frac{1}{10} - 4 \text{ arc tg } \frac{1}{515} - \text{arc tg } \frac{1}{239}$.

17. ***a***) Temos:

2 arc tg α_m = arc tg α_{m-1}, $m \geqslant 0$, $\alpha_m > 0$.

Daí,

arc tg α_m + arc tg α_m = arc tg α_{m-1}

$$\frac{\alpha_m + \alpha_m}{1 - \alpha_m^2} = \alpha_{m-1} \Rightarrow \alpha_{m-1} \cdot \alpha_m^2 + 2\alpha_m - \alpha_{m-1} = 0.$$

De $\alpha_m > 0$ segue

$$\alpha_m = \frac{-2 + \sqrt{4 + 4\,\alpha_{m-1}^2}}{2\,\alpha_{m-1}}, \text{ ou seja,}$$

$$\alpha_m = \frac{-1 + \sqrt{1 + \alpha_{m-1}^2}}{\alpha_{m-1}}.$$

b) De 2 arc tg α_m = arc tg α_{m-1} segue

2 arc tg α_1 = arc tg α_0. Mas $\alpha_0 = 1$. Logo, $\frac{\pi}{4} = 2 \text{ arc tg } \alpha_1$.

Temos 2 arc tg α_2 = arc tg α_1, 2 arc tg α_3 = arc tg α_2; ...; 2 arc tg α_m = arc tg α_{m-1}.
Portanto,

$$\frac{\pi}{4} = 2 \text{ arc tg } \alpha_1 = \underbrace{2 \cdot 2}_{2^2} \text{ arc tg } \alpha_2 = \underbrace{2 \cdot 2 \cdot 2}_{2^3} \text{ arc tg } \alpha_3 = \ldots$$

$= \ldots = 2^m$ arc tg α_m, ou seja,

$$\frac{\pi}{4} = 2^m \text{ arc tg } \alpha_m.$$

c) Pelo item a, $\alpha_m = \dfrac{-1+\sqrt{1+\alpha_{m-1}^2}}{\alpha_{m-1}} \; \dfrac{(1+\sqrt{1+\alpha_{m-1}^2}\,)}{(1+\sqrt{1+\alpha_{m-1}^2}\,)}$, ou seja,

$$\alpha_m = \frac{1+\alpha_{m-1}^2-1}{\alpha_{m-1}\,(1+\sqrt{1+\alpha_{m-1}^2}\,)}.$$

Logo,

$$\alpha_m = \frac{\alpha_{m-1}}{1+\sqrt{1+\alpha_{m-1}^2}} \leqslant \frac{\alpha_{m-1}}{2} \leqslant \frac{\alpha_{m-2}}{2\cdot 2} \leqslant \ldots \leqslant \frac{\alpha_0}{2^m}$$

Então,

$\alpha_m \leqslant \dfrac{\alpha_0}{2^m}$. Como $\alpha_0 = 1$, temos $\alpha_m \leqslant \dfrac{1}{2^m}$.

d) Pela fórmula de Gregory,

$$\text{arc tg}\,\alpha_m = \sum_{k=0}^{\infty} (-1)^k \frac{\alpha_m^{2k+1}}{2k+1} \quad ①$$

De b, $\dfrac{\pi}{4} = 2^m$ arc tg α_m ②

Substituindo ① em ② segue

$$\frac{\pi}{4} = 2^m \sum_{k=0}^{\infty} (-1)^k \frac{\alpha_m^{2k+1}}{2k+1}.$$

e) Do Exemplo 7, parte b, temos

$$\left|\text{arc tg}\,\alpha_m - \sum_{k=0}^{n} \frac{(-1)^k\,\alpha_m^{2k+1}}{2k+1}\right| \leqslant \frac{\alpha_m^{2n+3}}{2n+3}.$$

De d,

$$\left|\frac{\pi}{4} - 2^m \sum_{k=0}^{n} \frac{(-1)^k\,\alpha_m^{2k+1}}{2k+1}\right| \leqslant \frac{\alpha_m^{2n+3}}{2n+3}.$$

Ou ainda,

$$\left|\pi - 2^m \cdot 2^2 \sum_{k=0}^{n} \frac{(-1)^k\ \alpha_m^{2k+1}}{2k+1}\right| \leqslant \frac{4}{2n+3}\cdot \alpha_m^{2n+3}.$$

De c, $\alpha_m \leqslant \dfrac{1}{2^m} \Rightarrow (\alpha_m)^{2n+3} \leqslant \left(\dfrac{1}{2^m}\right)^{2n+3} \leqslant \left(\dfrac{1}{2^m}\right)^{2n+2}.$

Logo, $\alpha_m^{2n+3} \leqslant \dfrac{1}{2^{2m(n+1)}} = \dfrac{1}{4^{m(n+1)}}.$

Então,

$$\left|\pi - 2^{m+2} \sum_{k=0}^{n} (-1)^k\ \frac{\alpha_m^{2k+1}}{2k+1}\right| \leqslant \frac{4}{(2n+3)4^{m(n+1)}}$$

f) Em particular, fazendo $n = 0$ em (e),

$$\left|\pi - 2^{m+2}\ \alpha_m\right| \leqslant \frac{4}{3\cdot 4^m}.$$

Para interpretação geométrica veja 17*f* na seção de Respostas.

18. Vamos mostrar por indução.
Primeiro vamos mostrar que é válida para $p = 1$.
Temos

$$\frac{\pi^2}{6} = 1 + \sum_{k=1}^{\infty} \frac{1}{k^2(k+1)} = 1 + \sum_{k=1}^{\infty}\left[\frac{1}{k^2} - \frac{1}{k(k+1)}\right]$$

Como $\dfrac{\pi^2}{6} = \displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$ e $1 = \displaystyle\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$, segue

$$\frac{\pi^2}{6} - 1 = \underbrace{\sum_{k=1}^{\infty}\frac{1}{k^2}}_{\frac{\pi^2}{6}} - \underbrace{\sum_{k=1}^{\infty}\frac{1}{k(k+1)}}_{1}$$

Agora vamos mostrar que, sendo verdadeira para p, é verdadeira para $p + 1$.

Temos que $\dfrac{\pi^2}{6} = \displaystyle\sum_{k=1}^{p}\frac{1}{k^2} + p!\sum_{k=1}^{\infty} \frac{1}{k^2(k+1)(k+2)...(k+p)}$ ①

(é válida).

Por outro lado,

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)(k+2)...(k+p+1)} = \frac{1}{p+1} \sum_{k=1}^{\infty} (b_k - b_{k+1})$$

onde $b_k = \dfrac{1}{k(k+1)(k+2)...(k+p)}$ (série telescópica)

$b_1 = \dfrac{1}{(p+1)!}$ e $\lim\limits_{k\to\infty} b_k = 0.$

Portanto,

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)(k+2)...(k+p+1)} = \frac{1}{(p+1)(p+1)!} = \frac{1}{(p+1)^2\, p!}.$$

Daí,

$$\frac{\pi^2}{6} - p!\underbrace{\frac{1}{(p+1)^2\, p!}} = \frac{\pi^2}{6} - p!\sum_{k=1}^{\infty} \frac{1}{k(k+1)(k+2)...(k+p+1)} \quad ②$$

Substituindo no 2.º membro de ② o valor de $\dfrac{\pi^2}{6}$ considerado verdadeiro em ①, segue

$$\frac{\pi^2}{6} - \frac{1}{(p+1)^2} = \sum_{k=1}^{\infty} \frac{1}{k^2} + p!\sum_{k=1}^{\infty} \frac{1}{k^2(k+1)(k+2)...(k+p)} -$$

$$- p!\sum_{k=1}^{\infty} \frac{1}{k(k+1)(k+2)...(k+p+1)} =$$

$$= \sum_{k=1}^{\infty} \frac{1}{k^2} + p!\sum_{k=1}^{\infty}\left[\frac{1}{k^2\,(k+1)(k+2)...(k+p)} - \frac{1}{k(k+1)(k+2)...(k+p+1)}\right] =$$

$$= \sum_{k=1}^{\infty} \frac{1}{k^2} + p!\sum_{k=1}^{\infty} \frac{(p+1)}{k^2(k+1)(k+2)...(k+p+1)} =$$

$$= \sum_{k=1}^{\infty} \frac{1}{k^2} + \underbrace{p!\,(p+1)}_{(p+1)!}\sum_{k=1}^{\infty} \frac{1}{k^2(k+1)(k+2)...(k+p+1)}.$$

Portanto,

$$\frac{\pi^2}{6} = \sum_{k=1}^{p+1} \frac{1}{k^2} + (p+1)! \sum_{k=1}^{\infty} \frac{1}{k^2(k+1)(k+2)...(k+p+1)}$$

mostrando que a igualdade é válida para $p + 1$.

19. Vamos mostrar por indução.
Primeiro vamos mostrar que é válido para $p = 1$.
Fazendo $p = 1$ em

$$s = \sum_{k=1}^{p} \frac{1}{k}\left[\frac{\pi^2}{6} - \left(1 + \frac{1}{2^2} + \frac{1}{3^2} + ... + \frac{1}{k^2}\right)\right] + p! \sum_{k=1}^{\infty} \frac{1}{k^3(k+1)...(k+p)}$$

temos

$$s = \frac{\pi^2}{6} - 1 + \sum_{k=1}^{\infty} \frac{1}{k^3(k+1)} = \frac{\pi^2}{6} - 1 + \sum_{k=1}^{\infty}\left[\frac{1}{k^3} - \frac{1}{k^2(k+1)}\right] =$$

$$= \frac{\pi^2}{6} - 1 + \sum_{k=1}^{\infty} \frac{1}{k^3} - \sum_{k=1}^{\infty}\left[\frac{1}{k^2} - \frac{1}{k(k+1)}\right] =$$

$$= \frac{\pi^2}{6} - 1 + \sum_{k=1}^{\infty} \frac{1}{k^3} - \underbrace{\sum_{k=1}^{\infty} \frac{1}{k^2}}_{\frac{\pi^2}{6}} + \underbrace{\sum_{k=1}^{\infty} \frac{1}{k(k+1)}}_{1}$$

Portanto,

$$s = \sum_{k=1}^{\infty} \frac{1}{k^3} \text{ e a igualdade se verifica para } p = 1.$$

Agora, vamos mostrar que, sendo verdadeira para p, é verdadeira para $p + 1$.
Considerando

$$\sum_{k=1}^{\infty} \frac{1}{k^3} = \sum_{k=1}^{p} \frac{1}{k}\left[\frac{\pi^2}{6} - \left(1 + \frac{1}{2^2} + \frac{1}{3^2} + ... + \frac{1}{k^2}\right)\right] +$$

$$+ p! \sum_{k=1}^{\infty} \frac{1}{k^3(k+1)(k+2)...(k+p)} \qquad \text{① verdadeira.}$$

Por outro lado,

$$p!\sum_{k=1}^{\infty} \frac{1}{k^2(k+1)...(k+p+1)} = \frac{p!}{p+1}\sum_{k=1}^{\infty}\left[\frac{1}{k^2(k+1)...(k+p)} - \frac{1}{k(k+1)...(k+p+1)}\right] =$$

$$= \frac{1}{p+1}\underbrace{\left[p!\sum_{k=1}^{\infty}\frac{1}{k^2(k+1)...(k+p)}\right]}_{\frac{p^2}{6}-\sum_{k=1}^{p}\frac{1}{k^2}} - \frac{1}{p+1}\underbrace{\left[p!\sum_{k=1}^{\infty}\frac{1}{k(k+1)...(k+p+1)}\right]}_{\frac{1}{(p+1)^2\,p!}}$$

(do Exercício 18).

Portanto,

$$p!\sum_{k=1}^{\infty}\frac{1}{k^2(k+1)...(k+p+1)} = \frac{1}{p+1}\left(\frac{\pi^2}{6} - \sum_{k=1}^{p}\frac{1}{k^2}\right) - \frac{1}{(p+1)^3} =$$

$$= \frac{1}{(p+1)}\left[\frac{\pi^2}{6} - \sum_{k=1}^{p+1}\frac{1}{k^2}\right].$$

Somando e subtraindo esta quantidade ao 2.º membro da igualdade ① que se verifica, segue

$$\sum_{k=1}^{\infty}\frac{1}{k^3} = \sum_{k=1}^{p}\frac{1}{k}\left[\frac{\pi^2}{6} - \left(1+\frac{1}{2^2}+\frac{1}{3^2}+...+\frac{1}{k^2}\right)\right] +$$

$$+\frac{1}{(p+1)}\left[\frac{\pi^2}{6} - \sum_{k=1}^{p+1}\frac{1}{k^2}\right] + p!\sum_{k=1}^{\infty}\left[\frac{1}{k^3(k+1)(k+2)...(k+p)} - \frac{1}{k^2(k+1)...(k+p+1)}\right].$$

Resulta

$$\sum_{k=1}^{\infty}\frac{1}{k^3} = \sum_{k=1}^{p+1}\frac{1}{k}\left[\frac{\pi^2}{6} - \left(1+\frac{1}{2^2}+\frac{1}{3^2}+...+\frac{1}{k^2}\right)\right] +$$

$$+ p!\sum_{k=1}^{\infty}\frac{p+1}{k^3(k+1)(k+2)...(k+p+1)}$$

$$\sum_{k=1}^{\infty}\frac{1}{k^3} = \sum_{k=1}^{p+1}\frac{1}{k}\left[\frac{\pi^2}{6} - \left(1+\frac{1}{2^2}+\frac{1}{3^2}+...+\frac{1}{k^2}\right)\right] +$$

$$+ (p+1)!\sum_{k=1}^{\infty}\frac{1}{k^3(k+1)(k+2)...(k+p+1)},$$ mostrando que se verifica para $p+1$.

Como $\lim\limits_{p\to\infty} p!\sum\limits_{k=1} \dfrac{1}{k^3(k+1)(k+2)...(k+p)} = 0$, segue que

$$s = \sum_{k=1}^{\infty} \frac{1}{k}\left[\frac{\pi^2}{6} - \left(1 + \frac{1}{2^2} + ... + \frac{1}{k^2}\right)\right].$$

Observação: Pelo exercício anterior,

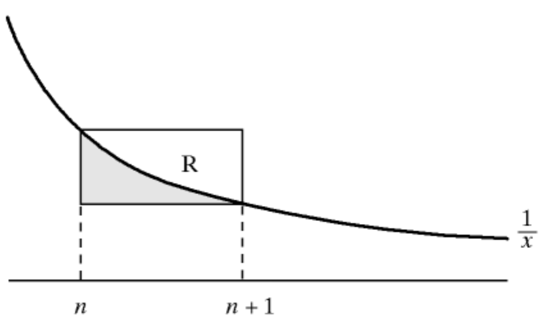
$$\lim_{p\to\infty} p!\sum_{k=1}^{\infty} \frac{1}{k^2(k+1)(k+2)...(k+p)} = 0.$$

Para todo $p \geqslant 1$,

$$\sum_{k=1}^{\infty} \frac{1}{k^3(k+1)(k+2)...(k+p)} \leqslant \sum_{k=1}^{\infty} \frac{1}{k^2(k+1)(k+2)...(k+p)}.$$

Logo, $\lim\limits_{p\to\infty} p!\sum\limits_{k=1}^{\infty} \dfrac{1}{k^3(k+1)(k+2)...(k+p)} = 0.$

20.



$$\ln(n+1) - \ln n - \frac{1}{n+1} < \frac{1}{2}\left[\frac{1}{n} - \frac{1}{n+1}\right], n \geqslant 1,$$

onde o primeiro membro é a área hachurada e o segundo membro a metade da área do retângulo R.

Temos então

$$\ln 2 - \ln 1 - \frac{1}{2} < \frac{1}{2}\left[\frac{1}{1} - \frac{1}{2}\right]$$

$$\ln 3 - \ln 2 - \frac{1}{3} < \frac{1}{2}\left[\frac{1}{2} - \frac{1}{3}\right]$$

$$\ldots$$

$$\ln(n+1) - \ln n - \frac{1}{n+1} < \frac{1}{2}\left[\frac{1}{n} - \frac{1}{n+1}\right].$$

Somando membro a membro, resulta, para $n \geqslant 1$,

$$\ln(n+1) - \sum_{k=1}^{n} \frac{1}{k+1} < \frac{1}{2}\left[\sum_{k=1}^{n} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{k+1}\right]$$

ou seja,

$$\ln(n+1) - \sum_{k=2}^{n+1} \frac{1}{k} < \frac{1}{2}\left[\sum_{k=2}^{n+1} \frac{1}{k-1} - \sum_{k=2}^{n+1} \frac{1}{k}\right]$$

Então, para $n \geqslant 2$,

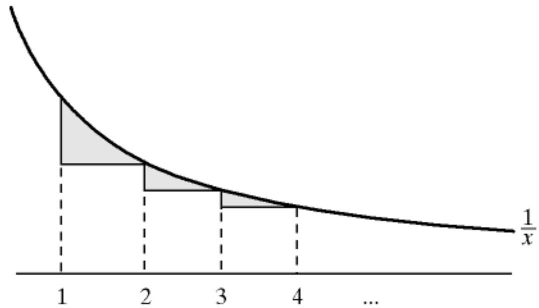
$$\ln n - \sum_{k=2}^{n} \frac{1}{k} < \frac{1}{2}\left[\sum_{k=2}^{n} \frac{1}{k-1} - \sum_{k=2}^{n} \frac{1}{k}\right]$$

ou ainda,

$$\ln n - \sum_{k=2}^{n} \frac{1}{k} < \frac{1}{2} \sum_{k=2}^{n} \frac{1}{k(k-1)},\ n \geqslant 2.$$

De $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k(k-1)} = \sum_{k=2}^{\infty}\left[\frac{1}{k-1} - \frac{1}{k}\right] = 1$, pois trata-se de uma série telescópica e, tendo em vista a desigualdade acima, resulta

$$\lim_{n \to \infty}\left[\ln n - \sum_{k=2}^{n} \frac{1}{k}\right] \leqslant \frac{1}{2}.$$



A soma das áreas reticuladas é menor que $\frac{1}{2}$.

21. ***a***) Temos $a_n = \dfrac{3 \cdot 5 \cdot 7 \cdot \ldots \cdot (2n+1)}{2 \cdot 4 \cdot 6 \cdot \ldots \cdot 2n} \dfrac{1}{\sqrt{n}}$

$$a_{n+1} = \frac{3 \cdot 5 \cdot 7 \cdot \ldots \cdot (2n+1) \cdot (2n+3)}{2 \cdot 4 \cdot 6 \cdot \ldots \cdot 2n \cdot (2n+2)} \frac{1}{\sqrt{n+1}}$$

Portanto,

$$\frac{a_{n+1}}{a_n} = \frac{2n+3}{2n+\ 2} \cdot \frac{\sqrt{n}}{\sqrt{n+1}} < 1 \Rightarrow a_{n+1} < a_n \Rightarrow a_n \text{ é decrescente.}$$

$$\left(\text{Veja: } \frac{a_{n+1}}{a_n} = \sqrt{\frac{(2n+3)^2\, n}{(2n+2)^2\ (n+1)}} = \sqrt{\frac{4n^3+12n^2+9n}{4n^3+12n^2+12n+4}} < 1,\ n \geqslant 1. \right)$$

b) $a_n = \dfrac{3}{2} \cdot \dfrac{5}{4} \cdot \dfrac{7}{6} \cdot \ldots \cdot \left(\dfrac{2n+1}{2n} \right) \cdot \dfrac{1}{\sqrt{n}} =$

$$= \left(1 + \frac{1}{2}\right) \cdot \left(1 + \frac{1}{4}\right) \cdot \left(1 + \frac{1}{6}\right) \cdot \ldots \cdot \left(1 + \frac{1}{2n}\right) \cdot \frac{1}{\sqrt{n}}.$$

$$\ln a_n = \ln \left[\left(1 + \frac{1}{2}\right) \cdot \left(1 + \frac{1}{4}\right) \cdot \left(1 + \frac{1}{6}\right) \cdot \ldots \cdot \left(1 + \frac{1}{2n}\right) \cdot \frac{1}{\sqrt{n}} \right]$$

$$\ln a_n = \ln\left(1 + \frac{1}{2}\right) + \ln\left(1 + \frac{1}{4}\right) + \ln\left(1 + \frac{1}{6}\right) + \ldots + \ln\left(1 + \frac{1}{2n}\right) + \underbrace{\ln \frac{1}{\sqrt{n}}}_{-\frac{1}{2}\ln n}$$

Tendo em vista a sugestão

$$\ln(1+x) > x - \frac{x^2}{2},\ x > 0,\ \text{vem}\ \ln\left(1+\frac{1}{2}\right) > \frac{1}{2} - \frac{1}{8};\ \ln\left(1+\frac{1}{4}\right) > \frac{1}{4} - \frac{1}{8\cdot 2^2};\ ...$$
$$...;\ \ln\left(1+\frac{1}{2n}\right) > \frac{1}{2n} - \frac{1}{8n^2}.$$

Portanto,

$$\ln a_n > \frac{1}{2} - \frac{1}{8} + \frac{1}{4} - \frac{1}{8\cdot 2^2} + \frac{1}{6} - \frac{1}{8\cdot 3^2} + ... + \frac{1}{2n} - \frac{1}{8n^2} - \frac{1}{2}\ln n,\ \text{ou seja,}$$
$$\ln a_n > \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n} - \ln n\right) - \frac{1}{8}\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + ... + \frac{1}{n^2}\right)\ \text{e, portanto,}$$
$$a_n > e^{\frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n} - \ln n\right) - \frac{1}{8}\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + ... + \frac{1}{n^2}\right)}$$

Observação:

Seja $g(x) = \ln(1+x) - x + \frac{x^2}{2},\ x > 0;\ g(0) = 0$ e

$g'(x) = \frac{1-(1+x)^2}{1+x} < 0,\ x > 0;$ então, $g(x)$ é decrescente em $[0, +\infty[$. Como $g(0) = 0$,

$g(x) < 0$ para $x > 0$, em ou seja, $\ln(1+x) < x - \frac{x^2}{2},\ x > 0.$

c) É só observar que $\lim\limits_{n\to\infty}\left[\ln n - \sum\limits_{k=2}^{n}\frac{1}{k}\right] \leqslant \frac{1}{2}$ (Exercício 20)

e $\lim\limits_{n\to\infty}\left[1 + \frac{1}{2^2} + ... + \frac{1}{n^2}\right] = \frac{\pi^2}{6}.$

Exercícios 2.2

2. ***a***) $\sum\limits_{k=1}^{\infty}\frac{1}{k^2} = \left[\frac{1}{1^2} + \frac{1}{2^2}\right] + \left[\frac{1}{3^2} + \frac{1}{4^2}\right] + ... + \left[\frac{1}{(2n-1)^2} + \frac{1}{(2n)^2}\right] + ...$

$$= \sum_{k=1}^{\infty}\left[\frac{1}{(2k-1)^2} + \frac{1}{(2k)^2}\right].$$

Temos, então,

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} + \sum_{k=1}^{\infty} \frac{1}{2^2 k^2}$$

pois as séries do segundo membro são convergentes. Segue que

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{3}{4} \sum_{k=1}^{\infty} \frac{1}{k^2}.$$

Tendo em vista $\displaystyle\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}$, resulta $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{4}{3} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}$.

b) $\displaystyle\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k^2} = \sum_{k=1}^{\infty} \left[\frac{1}{(2k-1)^2} - \frac{1}{(2k)^2} \right] = \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} - \frac{1}{4} \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{k^2}$.

Tendo em vista que $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$, resulta $\displaystyle\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k^2} = \frac{\pi^2}{12}$.

c) A desigualdade $\displaystyle\left| s - \sum_{k=1}^{n} (-1)^{k+1} \frac{1}{k^2} \right| \leqslant a_{n+1}$ estabelece uma avaliação para o módulo do erro que se comete ao aproximar s pela soma parcial s_n. Temos $\displaystyle a_{n+1} = \frac{1}{(n+1)^2}$.

Portanto,

$$\left| s - \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k^2} \right| \leqslant \frac{1}{(n+1)^2}.$$

Como $\displaystyle\frac{1}{(n+1)^2} < 10^{-3}$, temos $n \geqslant 31$.