Chapter 3

Problem Solutions

3.1 If a_o were to increase, the bandgap energy would decrease and the material would begin to behave less like a semiconductor and more like a metal. If a_o were to decrease, the bandgap energy would increase and the material would begin to behave more like an insulator.

3.2
Schrodinger's wave equation

$$\frac{-\hbar^2}{2m}\cdot\frac{\partial^2\Psi(x,t)}{\partial x^2}+V(x)\cdot\Psi(x,t)=j\hbar\frac{\partial\Psi(x,t)}{\partial t}$$

Let the solution be of the form

$$\Psi(x,t)=u(x)\exp\left[j\left(kx-\left(\frac{E}{\hbar}\right)t\right)\right]$$

Region I, $V(x)=0$, so substituting the proposed solution into the wave equation, we obtain

$$\frac{-\hbar^2}{2m}\cdot\frac{\partial}{\partial x}\left\{jku(x)\exp\left[j\left(kx-\left(\frac{E}{\hbar}\right)t\right)\right]+\frac{\partial u(x)}{\partial x}\exp\left[j\left(kx-\left(\frac{E}{\hbar}\right)t\right)\right]\right\}$$
$$=j\hbar\left(\frac{-jE}{\hbar}\right)\cdot u(x)\exp\left[j\left(kx-\left(\frac{E}{\hbar}\right)t\right)\right]$$

which becomes

$$\frac{-\hbar^2}{2m}\left\{(jk)^2u(x)\exp\left[j\left(kx-\left(\frac{E}{\hbar}\right)t\right)\right]+2jk\frac{\partial u(x)}{\partial x}\exp\left[j\left(kx-\left(\frac{E}{\hbar}\right)t\right)\right]+\frac{\partial^2u(x)}{\partial x^2}\exp\left[j\left(kx-\left(\frac{E}{\hbar}\right)t\right)\right]\right\}$$
$$=+Eu(x)\exp\left[j\left(kx-\left(\frac{E}{\hbar}\right)t\right)\right]$$

This equation can then be written as

$$-k^2u(x)+2jk\frac{\partial u(x)}{\partial x}+\frac{\partial^2u(x)}{\partial x^2}+\frac{2mE}{\hbar^2}\cdot u(x)=0$$

Setting $u(x)=u_1(x)$ for region I, this equation becomes

$$\frac{d^2u_1(x)}{dx^2}+2jk\frac{du_1(x)}{dx}-\left(k^2-\alpha^2\right)u_1(x)=0$$

where

$$\alpha^2=\frac{2mE}{\hbar^2}$$

Q.E.D.

In region II, $V(x)=V_O$. Assume the same form of the solution

$$\Psi(x,t)=u(x)\exp\left[j\left(kx-\left(\frac{E}{\hbar}\right)t\right)\right]$$

Substituting into Schrodinger's wave equation, we obtain

$$\frac{-\hbar^2}{2m}\left\{(jk)^2u(x)\exp\left[j\left(kx-\left(\frac{E}{\hbar}\right)t\right)\right]+2jk\frac{\partial u(x)}{\partial x}\exp\left[j\left(kx-\left(\frac{E}{\hbar}\right)t\right)\right]+\frac{\partial^2u(x)}{\partial x^2}\exp\left[j\left(kx-\left(\frac{E}{\hbar}\right)t\right)\right]\right\}$$
$$+V_Ou(x)\exp\left[j\left(kx-\left(\frac{E}{\hbar}\right)t\right)\right]$$
$$=Eu(x)\exp\left[j\left(kx-\left(\frac{E}{\hbar}\right)t\right)\right]$$

This equation can be written as

$$-k^2u(x)+2jk\frac{\partial u(x)}{\partial x}+\frac{\partial^2u(x)}{\partial x^2}-\frac{2mV_O}{\hbar^2}u(x)+\frac{2mE}{\hbar^2}u(x)=0$$

Setting $u(x)=u_2(x)$ for region II, this equation becomes

$$\frac{d^2u_2(x)}{dx^2}+2jk\frac{du_2(x)}{dx}-\left(k^2-\alpha^2+\frac{2mV_O}{\hbar^2}\right)u_2(x)=0$$

where

$$\alpha^2=\frac{2mE}{\hbar^2}$$

Q.E.D.

3.3
We have

$$\frac{d^2u_1(x)}{dx^2}+2jk\frac{du_1(x)}{dx}-\left(k^2-\alpha^2\right)u_1(x)=0$$

The proposed solution is

$$u_1(x)=A\exp\left[j(\alpha-k)x\right]+B\exp\left[-j(\alpha+k)x\right]$$

The first derivative is

$$\frac{du_1(x)}{dx}=j(\alpha-k)A\exp\left[j(\alpha-k)x\right]$$
$$-j(\alpha+k)B\exp\left[-j(\alpha+k)x\right]$$

and the second derivative becomes

$$\frac{d^2u_1(x)}{dx^2}=\left[j(\alpha-k)\right]^2A\exp\left[j(\alpha-k)x\right]$$
$$+\left[j(\alpha+k)\right]^2B\exp\left[-j(\alpha+k)x\right]$$

Substituting these equations into the differential equation, we find

$$-(\alpha-k)^2A\exp\left[j(\alpha-k)x\right]$$
$$-(\alpha+k)^2B\exp\left[-j(\alpha+k)x\right]$$
$$+2jk\{j(\alpha-k)A\exp\left[j(\alpha-k)x\right]$$
$$-j(\alpha+k)B\exp\left[-j(\alpha+k)x\right]\}$$
$$-\left(k^2-\alpha^2\right)\{A\exp\left[j(\alpha-k)x\right]$$
$$+B\exp\left[-j(\alpha+k)x\right]\}=0$$

Combining terms, we have

$$\left\{-\left(\alpha^2-2\alpha k+k^2\right)-2k(\alpha-k)\right.$$
$$\left.-\left(k^2-\alpha^2\right)\right\}A\exp\left[j(\alpha-k)x\right]$$
$$+\left\{-\left(\alpha^2+2\alpha k+k^2\right)+2k(\alpha+k)\right.$$
$$\left.-\left(k^2-\alpha^2\right)\right\}B\exp\left[-j(\alpha+k)x\right]=0$$

We find that

$0=0$ Q.E.D.

For the differential equation in $u_2(x)$ and the proposed solution, the procedure is exactly the same as above.

3.4
We have the solutions

$$u_1(x)=A\exp\left[j(\alpha-k)x\right]+B\exp\left[-j(\alpha+k)x\right]$$

for $0<x<a$

$$u_2(x)=C\exp\left[j(\beta-k)x\right]+D\exp\left[-j(\beta+k)x\right]$$

for $-b<x<0$

The boundary conditions:

$$u_1(0)=u_2(0)$$

which yields

$$A+B-C-D=0$$

Also

$$\left.\frac{du_1}{dx}\right|_{x=0}=\left.\frac{du_2}{dx}\right|_{x=0}$$

which yields

$$(\alpha-k)A-(\alpha+k)B-(\beta-k)C+(\beta+k)D=0$$

The third boundary condition is

$$u_1(a)=u_2(-b)$$

which gives

$$A\exp\left[j(\alpha-k)a\right]+B\exp\left[-j(\alpha+k)a\right]$$
$$=C\exp\left[j(\beta-k)(-b)\right]+D\exp\left[-j(\beta+k)(-b)\right]$$

This becomes

$$A\exp\left[j(\alpha-k)a\right]+B\exp\left[-j(\alpha+k)a\right]$$
$$-C\exp\left[-j(\beta-k)b\right]-D\exp\left[j(\beta+k)b\right]=0$$

The last boundary condition is

$$\left.\frac{du_1}{dx}\right|_{x=a}=\left.\frac{du_2}{dx}\right|_{x=-b}$$

which gives

$$j(\alpha-k)A\exp\left[j(\alpha-k)a\right]$$
$$-j(\alpha+k)B\exp\left[-j(\alpha+k)a\right]$$
$$=j(\beta-k)C\exp\left[j(\beta-k)(-b)\right]$$
$$-j(\beta+k)D\exp\left[-j(\beta+k)(-b)\right]$$

This becomes

$$(\alpha-k)A\exp\left[j(\alpha-k)a\right]$$
$$-(\alpha+k)B\exp\left[-j(\alpha+k)a\right]$$
$$-(\beta-k)C\exp\left[-j(\beta-k)b\right]$$
$$+(\beta+k)D\exp\left[j(\beta+k)b\right]=0$$

3.5 Computer plot

3.6 Computer plot

3.7

$$P'\frac{\sin\alpha a}{\alpha a}+\cos\alpha a=\cos ka$$

Let $ka=y$, $\alpha a=x$

Then

$$P'\frac{\sin x}{x}+\cos x=\cos y$$

Consider $\dfrac{d}{dy}$ of this function

$$\frac{d}{dy}\left\{\left[P'\cdot(x)^{-1}\cdot\sin x\right]+\cos x\right\}=-\sin y$$

We obtain

$$P'\left\{(-1)(x)^{-2}\sin x\frac{dx}{dy}+(x)^{-1}\cos x\frac{dx}{dy}\right\}$$

$$-\sin x\frac{dx}{dy}=-\sin y$$

Then

$$\frac{dx}{dy}\left\{P'\left[\frac{-1}{x^2}\sin x+\frac{\cos x}{x}\right]-\sin x\right\}=-\sin y$$

For $y = ka = n\pi$, $n = 0, 1, 2, \ldots$

$$\Rightarrow \sin y = 0$$

So that, in general, then

$$\frac{dx}{dy}=0=\frac{d(\alpha a)}{d(ka)}=\frac{d\alpha}{dk}$$

And

$$\alpha=\sqrt{\frac{2mE}{\hbar^2}}\Rightarrow\frac{d\alpha}{dk}=\frac{1}{2}\left(\frac{2mE}{\hbar^2}\right)^{-1/2}\left(\frac{2m}{\hbar^2}\right)\frac{dE}{dk}$$

This implies that

$$\underline{\frac{d\alpha}{dk}=0=\frac{dE}{dk}}\ \text{ for }\ \underline{k=\frac{n\pi}{a}}$$

3.8

$$f(\alpha a)=9\frac{\sin\alpha a}{\alpha a}+\cos\alpha a=\cos ka$$

(a) $ka = \pi \Rightarrow \cos ka = -1$

1st point: $\alpha a = \pi$: 2nd point: $\alpha a = 1.66\pi$

(2nd point by trial and error)

Now

$$\alpha a=a\sqrt{\frac{2mE}{\hbar^2}}\Rightarrow E=\left(\frac{\alpha a}{a}\right)^2\cdot\frac{\hbar^2}{2m}$$

So

$$E=\frac{(\alpha a)^2}{\left(5x10^{-10}\right)^2}\cdot\frac{\left(1.054x10^{-34}\right)^2}{2\left(9.11x10^{-31}\right)}\Rightarrow$$

$$E=(\alpha a)^2\left[2.439x10^{-20}\right]\ (J)$$

or

$$E=(\alpha a)^2(0.1524)\quad(eV)$$

So

$$\alpha a=\pi\Rightarrow E_1=1.504\ eV$$

$$\alpha a=1.66\pi\Rightarrow E_2=4.145\ eV$$

Then

$$\underline{\Delta E=2.64\ eV}$$

(b)
$ka = 2\pi \Rightarrow \cos ka = +1$

1st point: $\alpha a = 2\pi$

2nd point: $\alpha a = 2.54\pi$

Then

$$E_3 = 6.0165\ eV$$

$$E_4 = 9.704\ eV$$

so

$$\underline{\Delta E=3.69\ eV}$$

(c)
$ka = 3\pi \Rightarrow \cos ka = -1$

1st point: $\alpha a = 3\pi$

2nd point: $\alpha a = 3.44\pi$

Then

$$E_5 = 13.537\ eV$$

$$E_6 = 17.799\ eV$$

so

$$\underline{\Delta E=4.26\ eV}$$

(d)
$ka = 4\pi \Rightarrow \cos ka = +1$

1st point: $\alpha a = 4\pi$

2nd point: $\alpha a = 4.37\pi$

Then

$$E_7 = 24.066\ eV$$

$$E_8 = 28.724\ eV$$

so

$$\underline{\Delta E=4.66\ eV}$$

3.9

(a) $0 < ka < \pi$

For $ka = 0 \Rightarrow \cos ka = +1$

By trial and error: 1st point: $\alpha a = 0.822\pi$

2nd point: $\alpha a = \pi$

From Problem 3.8, $E = (\alpha a)^2(0.1524) \quad (eV)$

Then

$$E_1 = 1.0163\ eV$$

$$E_2 = 1.5041\ eV$$

so

$$\underline{\Delta E=0.488\ eV}$$

(b)
$\pi < ka < 2\pi$

Using results of Problem 3.8

1st point: $\alpha a = 1.66\pi$

2nd point: $\alpha a = 2\pi$

Then

$E_3 = 4.145\ eV$

$E_4 = 6.0165\ eV$

so

$\Delta E = 1.87\ eV$

(c)
$2\pi < ka < 3\pi$

1st point: $\alpha a = 2.54\pi$

2nd point: $\alpha a = 3\pi$

Then

$E_5 = 9.704\ eV$

$E_6 = 13.537\ eV$

so

$\Delta E = 3.83\ eV$

(d)
$3\pi < ka < 4\pi$

1st point: $\alpha a = 3.44\pi$

2nd point: $\alpha a = 4\pi$

Then

$E_7 = 17.799\ eV$

$E_8 = 24.066\ eV$

so

$\Delta E = 6.27\ eV$

3.10

$$6\frac{\sin \alpha a}{\alpha a} + \cos \alpha a = \cos ka$$

Forbidden energy bands

(a) $ka = \pi \Rightarrow \cos ka = -1$

1st point: $\alpha a = \pi$

2nd point: $\alpha a = 1.56\pi$ (By trial and error)

From Problem 3.8, $E = (\alpha a)^2 (0.1524)\ eV$

Then

$E_1 = 1.504\ eV$

$E_2 = 3.660\ eV$

so

$\Delta E = 2.16\ eV$

(b)
$ka = 2\pi \Rightarrow \cos ka = +1$

1st point: $\alpha a = 2\pi$

2nd point: $\alpha a = 2.42\pi$

Then

$E_3 = 6.0165\ eV$

$E_4 = 8.809\ eV$

so

$\Delta E = 2.79\ eV$

(c)
$ka = 3\pi \Rightarrow \cos ka = -1$

1st point: $\alpha a = 3\pi$

2nd point: $\alpha a = 3.33\pi$

Then

$E_5 = 13.537\ eV$

$E_6 = 16.679\ eV$

so

$\Delta E = 3.14\ eV$

(d)
$ka = 4\pi \Rightarrow \cos ka = +1$

1st point: $\alpha a = 4\pi$

2nd point: $\alpha a = 4.26\pi$

Then

$E_7 = 24.066\ eV$

$E_8 = 27.296\ eV$

so

$\Delta E = 3.23\ eV$

3.11

Allowed energy bands

Use results from Problem 3.10.

(a)
$0 < ka < \pi$

1st point: $\alpha a = 0.759\pi$ (By trial and error)

2nd point: $\alpha a = \pi$

We have

$$E = (\alpha a)^2 (0.1524)\ eV$$

Then

$E_1 = 0.8665\ eV$

$E_2 = 1.504\ eV$

so

$\Delta E = 0.638\ eV$

(b)
$\pi < ka < 2\pi$

1st point: $\alpha a = 1.56\pi$

2nd point: $\alpha a = 2\pi$

Then

$E_3 = 3.660\ eV$

$E_4 = 6.0165\ eV$

so

$\Delta E = 2.36\ eV$

(c)
$2\pi < ka < 3\pi$

1st point: $\alpha a = 2.42\pi$

2nd point: $\alpha a = 3\pi$

Then

$E_5 = 8.809\ eV$

$E_6 = 13.537\ eV$

so

$$\underline{\Delta E = 4.73\ eV}$$

(d)

$3\pi < ka < 4\pi$

1st point: $\alpha a = 3.33\pi$

2nd point: $\alpha a = 4\pi$

Then

$E_7 = 16.679\ eV$

$E_8 = 24.066\ eV$

so

$$\underline{\Delta E = 7.39\ eV}$$

3.12

$$T = 100K\ ;\ \ E_g = 1.170 - \frac{(4.73x10^{-4})(100)^2}{636+100} \Rightarrow$$

$$\underline{E_g = 1.164\ eV}$$

$$T = 200K \Rightarrow \underline{E_g = 1.147\ eV}$$

$$T = 300K \Rightarrow \underline{E_g = 1.125\ eV}$$

$$T = 400K \Rightarrow \underline{E_g = 1.097\ eV}$$

$$T = 500K \Rightarrow \underline{E_g = 1.066\ eV}$$

$$T = 600K \Rightarrow \underline{E_g = 1.032\ eV}$$

3.13

The effective mass is given by

$$m^* = \left(\frac{1}{\hbar^2}\cdot\frac{d^2E}{dk^2}\right)^{-1}$$

We have that

$$\frac{d^2E}{dk^2}(curve\ A) > \frac{d^2E}{dk^2}(curve\ B)$$

so that

$$\underline{m^*(curve\ A) < m^*(curve\ B)}$$

3.14

The effective mass for a hole is given by

$$m_p^* = \left(\frac{1}{\hbar^2}\cdot\left|\frac{d^2E}{dk^2}\right|\right)^{-1}$$

We have that

$$\left|\frac{d^2E}{dk^2}\right|(curve\ A) > \left|\frac{d^2E}{dk^2}\right|(curve\ B)$$

so that

$$\underline{m_p^*(curve\ A) < m_p^*(curve\ B)}$$

3.15

Points A, B: $\dfrac{\partial E}{\partial k} < 0 \Rightarrow$ velocity in –x direction;

Points C, D: $\dfrac{\partial E}{\partial x} > 0 \Rightarrow$ velocity in +x direction;

Points A, D; $\dfrac{\partial^2 E}{\partial k^2} < 0 \Rightarrow$ negative effective mass;

Points B, C; $\dfrac{\partial^2 E}{\partial k^2} > 0 \Rightarrow$ positive effective mass;

3.16

$$E - E_C = \frac{k^2\hbar^2}{2m}$$

At $k = 0.1\left(\overset{\circ}{A}\right)^{-1} \Rightarrow \dfrac{1}{k} = 10\ \overset{\circ}{A} = 10^{-9}\ m$

So

$k = 10^{+9}\ m^{-1}$

For A:

$$(0.07)(1.6x10^{-19}) = \frac{(10^9)^2(1.054x10^{-34})^2}{2m}$$

which yields

$$m = 4.96x10^{-31}\ kg$$

so

$$\underline{curve\ A;\ \frac{m}{m_o} = 0.544}$$

For B:

$$(0.7)(1.6x10^{-19}) = \frac{(10^9)(1.054x10^{-34})^2}{2m}$$

which yields

$m = 4.96x10^{-32}\ kg$

so

$$\underline{Curve\ B:\ \frac{m}{m_o} = 0.0544}$$

3.17

$$E_V - E = \frac{k^2\hbar^2}{2m}$$

$k = 0.1\left(A^*\right)^{-1} \Rightarrow 10^9 \ m^{-1}$

For Curve A:

$$(0.08)\left(1.6x10^{-19}\right) = \frac{\left(10^9\right)^2\left(1.054x10^{-34}\right)^2}{2m}$$

which yields

$$m = 4.34x10^{-31} \ kg \Rightarrow \frac{m}{m_o} = 0.476$$

For Curve B:

$$(0.4)\left(1.6x10^{-19}\right) = \frac{\left(10^9\right)^2\left(1.054x10^{-34}\right)^2}{2m}$$

which yields

$$m = 8.68x10^{-32} \ kg \Rightarrow \frac{m}{m_o} = 0.0953$$

3.18

(a) $E = h\nu$

Then

$$\nu = \frac{E}{h} = \frac{(1.42)\left(1.6x10^{-19}\right)}{\left(6.625x10^{-34}\right)} \Rightarrow$$

$$\nu = 3.43x10^{14} \ Hz$$

(b)

$$\lambda = \frac{c}{\nu} = \frac{3x10^8}{3.43x10^{14}} = 8.75x10^{-7} \ m$$

or

$$\lambda = 0.875 \ \mu m$$

3.19

(c) Curve A: Effective mass is a constant
Curve B: Effective mass is positive around $k = 0$, and is negative around $k = \pm\frac{\pi}{2}$.

3.20

$$E = E_O - E_1 \cos\left[\alpha\left(k - k_O\right)\right]$$

$$\frac{dE}{dk} = \left(-E_1\right)(-\alpha)\sin\left[\alpha\left(k - k_O\right)\right]$$

$$= +E_1\alpha \sin\left[\alpha\left(k - k_O\right)\right]$$

So

$$\frac{d^2E}{dk^2} = E_1\alpha^2 \cos\left[\alpha\left(k - k_O\right)\right]$$

Then

$$\left.\frac{d^2E}{dk^2}\right|_{k=k_O} = E_1\alpha^2$$

We have

$$\frac{1}{m^*} = \frac{1}{\hbar^2}\cdot\frac{d^2E}{dk^2} = \frac{E_1\alpha^2}{\hbar^2}$$

or

$$m^* = \frac{\hbar^2}{E_1\alpha^2}$$

3.21

For the 3-dimensional infinite potential well, $V(x) = 0$ when $0 < x < a$, $0 < y < a$, and $0 < z < a$. In this region, the wave equation is

$$\frac{\partial^2\psi(x,y,z)}{\partial x^2} + \frac{\partial^2\psi(x,y,z)}{\partial y^2} + \frac{\partial^2\psi(x,y,z)}{\partial z^2} + \frac{2mE}{\hbar^2}\psi(x,y,z) = 0$$

Use separation of variables technique, so let

$$\psi(x,y,z) = X(x)Y(y)Z(z)$$

Substituting into the wave equation, we have

$$YZ\frac{\partial^2 X}{\partial x^2} + XZ\frac{\partial^2 Y}{\partial y^2} + XY\frac{\partial^2 Z}{\partial z^2} + \frac{2mE}{\hbar^2}\cdot XYZ = 0$$

Dividing by XYZ, we obtain

$$\frac{1}{X}\cdot\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\cdot\frac{\partial^2 Y}{\partial y^2} + \frac{1}{Z}\cdot\frac{\partial^2 Z}{\partial z^2} + \frac{2mE}{\hbar^2} = 0$$

Let

$$\frac{1}{X}\cdot\frac{\partial^2 X}{\partial x^2} = -k_x^2 \Rightarrow \frac{\partial^2 X}{\partial x^2} + k_x^2 X = 0$$

The solution is of the form

$$X(x) = A\sin k_x x + B\cos k_x x$$

Since $\psi(x,y,z) = 0$ at $x = 0$, then $X(0) = 0$ so that $B \equiv 0$.

Also, $\psi(x,y,z) = 0$ at $x = a$, then $X(a) = 0$ so we must have $k_x a = n_x\pi$, where $n_x = 1, 2, 3, \ldots$

Similarly, we have

$$\frac{1}{Y}\cdot\frac{\partial^2 Y}{\partial y^2} = -k_y^2 \quad \text{and} \quad \frac{1}{Z}\cdot\frac{\partial^2 Z}{\partial z^2} = -k_z^2$$

From the boundary conditions, we find

$k_y a = n_y \pi$ and $k_z a = n_z \pi$

where $n_y = 1, 2, 3, \ldots$ and $n_z = 1, 2, 3, \ldots$

From the wave equation, we have

$$-k_x^2 - k_y^2 - k_z^2 + \frac{2mE}{\hbar^2} = 0$$

The energy can then be written as

$$E = \frac{\hbar^2}{2m}\left(n_x^2 + n_y^2 + n_z^2\right)\left(\frac{\pi}{a}\right)^2$$

3.22

The total number of quantum states in the 3-dimensional potential well is given (in k-space) by

$$g_T(k)dk = \frac{\pi k^2 dk}{\pi^3} \cdot a^3$$

where

$$k^2 = \frac{2mE}{\hbar^2}$$

We can then write

$$k = \frac{1}{\hbar} \cdot \sqrt{2mE}$$

Taking the differential, we obtain

$$dk = \frac{1}{\hbar} \cdot \sqrt{2m} \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{E}} \cdot dE = \frac{1}{\hbar} \cdot \sqrt{\frac{m}{2E}} \cdot dE$$

Substituting these expressions into the density of states function, we obtain

$$g_T(E)dE = \frac{\pi a^3}{\pi^3}\left(\frac{2mE}{\hbar^2}\right) \cdot \frac{1}{\hbar} \cdot \sqrt{\frac{m}{2E}} \cdot dE$$

Noting that

$$\hbar = \frac{h}{2m}$$

this density of states function can be simplified and written as

$$g_T(E)dE = \frac{4\pi a^3}{h^3}(2m)^{3/2} \cdot \sqrt{E} \cdot dE$$

Dividing by a^3 will yield the density of states, so that

$$g(E) = \frac{4\pi (2m)^{3/2}}{h^3} \cdot \sqrt{E}$$

3.23

$$g_C(E) = \frac{4\pi\left(2m_n^*\right)^{3/2}}{h^3}\sqrt{E - E_C}$$

Now

$$g_T = \frac{4\pi\left(2m_n^*\right)^{3/2}}{h^3}\int_{E_C}^{E_C+kT}\sqrt{E - E_C} \cdot dE$$

$$= \frac{4\pi\left(2m_n^*\right)^{3/2}}{h^3}\left(\frac{2}{3}\right)\left(E - E_C\right)^{3/2}\Big|_{E_C}^{E_C+kT}$$

$$= \frac{4\pi\left(2m_n^*\right)^{3/2}}{h^3}\left(\frac{2}{3}\right)(kT)^{3/2}$$

Then

$$g_T = \frac{4\pi\left[2(0.067)\left(9.11x10^{-31}\right)\right]^{3/2}}{\left(6.625x10^{-34}\right)^3}\left(\frac{2}{3}\right)$$

$$\times\left[(0.0259)\left(1.6x10^{-19}\right)\right]^{3/2}$$

or

$$g_T = 3.28x10^{23}\ m^{-3} = 3.28x10^{17}\ cm^{-3}$$

3.24

$$g_V(E) = \frac{4\pi\left(2m_p^*\right)^{3/2}}{h^3}\sqrt{E_V - E}$$

Now

$$g_T = \frac{4\pi\left(2m_p^*\right)^{3/2}}{h^3}\int_{E_V-kT}^{E_V}\sqrt{E_V - E} \cdot dE$$

$$= \frac{4\pi\left(2m_p^*\right)^{3/2}}{h^3}\left(\frac{-2}{3}\right)\left(E_V - E\right)^{3/2}\Big|_{E_V-kT}^{E_V}$$

$$= \frac{4\pi\left(2m_p^*\right)^{3/2}}{h^3}\left(\frac{2}{3}\right)(kT)^{3/2}$$

$$g_T = \frac{4\pi\left[2(0.48)\left(9.11x10^{-31}\right)\right]^{3/2}}{\left(6.625x10^{-34}\right)^3}\left(\frac{2}{3}\right)$$

$$\times\left[(0.0259)\left(1.6x10^{-19}\right)\right]^{3/2}$$

or

$$g_T = 6.29x10^{24}\ m^{-3} = 6.29x10^{18}\ cm^{-3}$$

3.25

(a) $$g_C(E) = \frac{4\pi\left(2m_n^*\right)^{3/2}}{h^3}\sqrt{E - E_C}$$

$$= \frac{4\pi\left[2(1.08)\left(9.11x10^{-31}\right)\right]^{3/2}}{\left(6.625x10^{-34}\right)^3}\left(1.6x10^{-19}\right)^{1/2}\sqrt{E - E_C}$$

$$= 4.77x10^{46}\sqrt{E - E_C}\quad m^{-3}J^{-1}$$

or

$$g_C(E) = 7.63x10^{21}\sqrt{E - E_C}\ cm^{-3}eV^{-1}$$

Then

E	g_C
$E_C + 0.05\ eV$	$1.71x10^{21}\ cm^{-3}eV^{-1}$
$E_C + 0.10\ eV$	$2.41x10^{21}$
$E_C + 0.15\ eV$	$2.96x10^{21}$
$E_C + 0.20\ eV$	$3.41x10^{21}$

(b) $g_V(E) = \dfrac{4\pi(2m_p^*)^{3/2}}{h^3}\sqrt{E_V - E}$

$$= \frac{4\pi[2(0.56)(9.11x10^{-31})]^{3/2}}{(6.625x10^{-34})^3}(1.6x10^{-19})^{1/2}\sqrt{E_V - E}$$

$$= 1.78x10^{46}\sqrt{E_V - E}\quad m^{-3}J^{-1}$$

$$g_V(E) = 2.85x10^{21}\sqrt{E_V - E}\ cm^{-3}eV^{-1}$$

E	$g_V(E)$
$E_V - 0.05\ eV$	$0.637x10^{21}\ cm^{-3}\ eV^{-1}$
$E_V - 0.10\ eV$	$0.901x10^{21}$
$E_V - 0.15\ eV$	$1.10x10^{21}$
$E_V - 0.20\ eV$	$1.27x10^{21}$

3.26

$$\frac{g_C}{g_V} = \frac{(m_n^*)^{3/2}}{(m_p^*)^{3/2}} \Rightarrow \underline{\frac{g_C}{g_V} = \left(\frac{m_n^*}{m_p^*}\right)^{3/2}}$$

3.27

Computer Plot

3.28

$$\frac{g_i!}{N_i!(g_i - N_i)!} = \frac{10!}{8!(10-8)!}$$

$$= \frac{(10)(9)(8!)}{(8!)(2!)} = \frac{(10)(9)}{(2)(1)} \Rightarrow \underline{= 45}$$

3.29

(a) $f(E) = \dfrac{1}{1 + \exp\left[\dfrac{(E_C + kT) - E_C}{kT}\right]}$

$$= \frac{1}{1 + \exp(1)} \Rightarrow \underline{f(E) = 0.269}$$

(b)

$$1 - f(E) = 1 - \frac{1}{1 + \exp\left[\dfrac{(E_V - kT) - E_V}{kT}\right]}$$

$$= 1 - \frac{1}{1 + \exp(-1)} \Rightarrow \underline{1 - f(E) = 0.269}$$

3.30

$$f(E) = \frac{1}{1 + \exp\left(\dfrac{E - E_F}{kT}\right)}$$

(a) $E - E_F = kT$, $f(E) = \dfrac{1}{1 + \exp(1)} \Rightarrow$

$\underline{f(E) = 0.269}$

(b) $E - E_F = 5kT$, $f(E) = \dfrac{1}{1 + \exp(5)} \Rightarrow$

$\underline{f(E) = 6.69x10^{-3}}$

(c) $E - E_F = 10kT$, $f(E) = \dfrac{1}{1 + \exp(10)} \Rightarrow$

$\underline{f(E) = 4.54x10^{-5}}$

3.31

$$1 - f(E) = 1 - \frac{1}{1 + \exp\left(\dfrac{E - E_F}{kT}\right)}$$

or

$$1 - f(E) = \frac{1}{1 + \exp\left(\dfrac{E_F - E}{kT}\right)}$$

(a) $E_F - E = kT$, $\underline{1 - f(E) = 0.269}$

(b) $E_F - E = 5kT$, $\underline{1 - f(E) = 6.69x10^{-3}}$

(c) $E_F - E = 10kT$, $\underline{1 - f(E) = 4.54x10^{-5}}$

3.32

(a) $T = 300K \Rightarrow kT = 0.0259\ eV$

$$f(E) = \frac{1}{1 + \exp\left(\dfrac{E - E_F}{kT}\right)} \approx \exp\left[\frac{-(E - E_F)}{kT}\right]$$

E	$f(E)$
E_C	$6.43x10^{-5}$
$E_C + (1/2)kT$	$3.90x10^{-5}$
$E_C + kT$	$2.36x10^{-5}$
$E_C + (3/2)kT$	$1.43x10^{-5}$
$E_C + 2kT$	$0.87x10^{-5}$

(b) $T = 400K \Rightarrow kT = 0.03453$

E	$f(E)$
E_C	$7.17x10^{-4}$
$E_C + (1/2)kT$	$4.35x10^{-4}$
$E_C + kT$	$2.64x10^{-4}$
$E_C + (3/2)kT$	$1.60x10^{-4}$
$E_C + 2kT$	$0.971x10^{-4}$

3.33

$$E_n = \frac{\hbar^2 n^2 \pi^2}{2ma^2} = \frac{\left(1.054x10^{-34}\right)^2 n^2 \pi^2}{2\left(9.11x10^{-31}\right)\left(10x10^{-10}\right)^2}$$

or

$$E_n = 6.018x10^{-20} n^2 \ J = 0.376 n^2 \ eV$$

For $n = 4 \Rightarrow E_4 = 6.02\ eV$,

For $n = 5 \Rightarrow E_5 = 9.40\ eV$.

As a 1st approximation for $T > 0$, assume the probability of $n = 5$ state being occupied is the same as the probability of $n = 4$ state being empty. Then

$$1 - \frac{1}{1 + \exp\left(\frac{E_4 - E_F}{kT}\right)} = \frac{1}{1 + \exp\left(\frac{E_5 - E_F}{kT}\right)}$$

$$\Rightarrow \frac{1}{1 + \exp\left(\frac{E_F - E_4}{kT}\right)} = \frac{1}{1 + \exp\left(\frac{E_5 - E_F}{kT}\right)}$$

or

$$E_F - E_4 = E_5 - E_F \Rightarrow E_F = \frac{E_4 + E_5}{2}$$

Then

$$E_F = \frac{6.02 + 9.40}{2} \Rightarrow \underline{E_F = 7.71\ eV}$$

3.34

(a) For 3-Dimensional infinite potential well,

$$E = \frac{\hbar^2 \pi^2}{2ma^2}\left(n_x^2 + n_y^2 + n_z^2\right)$$

$$= \frac{\left(1.054x10^{-34}\right)^2 \pi^2}{2\left(9.11x10^{-31}\right)\left(10^{-9}\right)^2}\left(n_x^2 + n_y^2 + n_z^2\right)$$

$$= 0.376\left(n_x^2 + n_y^2 + n_z^2\right)\ eV$$

For 5 electrons, energy state corresponding to $n_x n_y n_z = 221 = 122$ contains both an electron and an empty state, so

$$E_F = (0.376)\left(2^2 + 2^2 + 1^2\right) \Rightarrow$$

$$\underline{E_F = 3.384\ eV}$$

(b) For 13 electrons, energy state corresponding to $n_x n_y n_z = 323 = 233$ contains both an electron and an empty state, so

$$E_F = (0.376)\left(2^2 + 3^2 + 3^2\right) \Rightarrow$$

$$\underline{E_F = 8.272\ eV}$$

3.35

The probability of a state at $E_1 = E_F + \Delta E$ being occupied is

$$f_1(E_1) = \frac{1}{1 + \exp\left(\frac{E_1 - E_F}{kT}\right)} = \frac{1}{1 + \exp\left(\frac{\Delta E}{kT}\right)}$$

The probability of a state at $E_2 = E_F - \Delta E$ being empty is

$$1 - f_2(E_2) = 1 - \frac{1}{1 + \exp\left(\frac{E_2 - E_F}{kT}\right)}$$

$$= 1 - \frac{1}{1 + \exp\left(\frac{-\Delta E}{kT}\right)} = \frac{\exp\left(\frac{-\Delta E}{kT}\right)}{1 + \exp\left(\frac{-\Delta E}{kT}\right)}$$

or

$$1 - f_2(E_2) = \frac{1}{1 + \exp\left(\frac{+\Delta E}{kT}\right)}$$

Hence, we have that

$$f_1(E_1) = 1 - f_2(E_2) \quad \text{Q.E.D.}$$

3.36

(a) At energy E_1, we want

$$\frac{\dfrac{1}{\exp\left(\dfrac{E_1-E_F}{kT}\right)}-\dfrac{1}{1+\exp\left(\dfrac{E_1-E_F}{kT}\right)}}{\dfrac{1}{1+\exp\left(\dfrac{E_1-E_F}{kT}\right)}}=0.01$$

This expression can be written as

$$\frac{1+\exp\left(\dfrac{E_1-E_F}{kT}\right)}{\exp\left(\dfrac{E_1-E_F}{kT}\right)}-1=0.01$$

$$\Rightarrow 1=(0.01)\exp\left(\frac{E_1-E_F}{kT}\right)$$

or

$$E_1=E_F+kT\ln(100)$$

Then

$$\underline{E_1=E_F+4.6kT}$$

(b)

At $E_1=E_F+4.6kT$,

$$f(E_1)=\frac{1}{1+\exp\left(\dfrac{E_1-E_F}{kT}\right)}=\frac{1}{1+\exp\left(\dfrac{4.6kT}{kT}\right)}$$

which yields

$$\underline{f(E_1)=0.00990\approx 0.01}$$

3.37

(a) $E_F=6.25\ eV$, $T=300K$, At $E=6.50\ eV$

$$f(E)=\frac{1}{1+\exp\left(\dfrac{6.50-6.25}{0.0259}\right)}=6.43x10^{-5}$$

or

$$\underline{6.43x10^{-3}\%}$$

(b)

$$T=950K\Rightarrow kT=(0.0259)\left(\frac{950}{300}\right)$$

or

$$kT=0.0820\ eV$$

Then

$$f(E)=\frac{1}{1+\exp\left(\dfrac{6.50-6.25}{0.0820}\right)}=0.0453$$

or $\underline{4.53\%}$

(c) $$1-0.01=\frac{1}{1+\exp\left(\dfrac{-0.30}{kT}\right)}=0.99$$

Then

$$1+\exp\left(\frac{-0.30}{kT}\right)=\frac{1}{0.99}=1.0101$$

which can be written as

$$\exp\left(\frac{+0.30}{kT}\right)=\frac{1}{0.0101}=99$$

Then

$$\frac{0.30}{kT}=\ln(99)\Rightarrow kT=\frac{0.30}{\ln(99)}=0.06529$$

So

$$\underline{T=756K}$$

3.38

(a)

$$f(E)=\frac{1}{1+\exp\left(\dfrac{7.15-7.0}{0.0259}\right)}=0.00304$$

or

$$\underline{0.304\%}$$

(b)

At $T=1000K\Rightarrow kT=0.08633\ eV$

Then

$$f(E)=\frac{1}{1+\exp\left(\dfrac{7.15-7.0}{0.08633}\right)}=0.1496$$

or $\underline{14.96\%}$

(c)

$$f(E)=\frac{1}{1+\exp\left(\dfrac{6.85-7.0}{0.0259}\right)}=0.997$$

or

$$\underline{99.7\%}$$

(d)

At $E=E_F$, $\underline{f(E)=\frac{1}{2}}$ for all temperatures.

3.39

For $E=E_1$,

$$f(E)=\frac{1}{1+\exp\left(\dfrac{E_1-E_F}{kT}\right)}\approx\exp\left[\frac{-(E_1-E_F)}{kT}\right]$$

Then

$$f(E_1) = \exp\left(\frac{-0.30}{0.0259}\right) \Rightarrow \underline{f(E_1) = 9.3x10^{-6}}$$

For $E = E_2$, $E_F - E_2 = 1.12 - 0.3 = 0.82\ eV$

$$1 - f(E) = 1 - \frac{1}{1 + \exp\left(\frac{-0.82}{0.0259}\right)}$$

or

$$1 - f(E) \approx 1 - \left[1 - \exp\left(\frac{-0.82}{0.0259}\right)\right]$$

$$= \exp\left(\frac{-0.82}{0.0259}\right) \Rightarrow \underline{1 - f(E) = 1.78x10^{-14}}$$

(b)
For $E_F - E_2 = 0.4 \Rightarrow E_1 - E_F = 0.72\ eV$
At $E = E_1$,

$$f(E) = \exp\left[\frac{-(E_1 - E_F)}{kT}\right] = \exp\left(\frac{-0.72}{0.0259}\right)$$

so

$$\underline{f(E) = 8.45x10^{-13}}$$

At $E = E_2$,

$$1 - f(E) = \exp\left[\frac{-(E_F - E_2)}{kT}\right] = \exp\left(\frac{-0.4}{0.0259}\right)$$

so

$$\underline{1 - f(E) = 1.96x10^{-7}}$$

3.40
(a) At $E = E_1$,

$$f(E) = \exp\left[\frac{-(E - E_F)}{kT}\right] = \exp\left(\frac{-0.30}{0.0259}\right)$$

or

$$\underline{f(E) = 9.3x10^{-6}}$$

At $E = E_2$, then
$E_F - E_2 = 1.42 - 0.3 = 1.12\ eV$,
So

$$1 - f(E) = \exp\left[\frac{-(E_F - E_2)}{kT}\right] = \exp\left(\frac{-1.12}{0.0259}\right)$$

or

$$\underline{1 - f(E) = 1.66x10^{-19}}$$

(b)
For $E_F - E_2 = 0.4 \Rightarrow E_1 - E_F = 1.02\ eV$,
At $E = E_1$,

$$f(E) = \exp\left[\frac{-(E_1 - E_F)}{kT}\right] = \exp\left(\frac{-1.02}{0.0259}\right)$$

or

$$\underline{f(E) = 7.88x10^{-18}}$$

At $E = E_2$,

$$1 - f(E) = \exp\left[\frac{-(E_F - E_2)}{kT}\right] = \exp\left(\frac{-0.4}{0.0259}\right)$$

or

$$\underline{1 - f(E) = 1.96x10^{-7}}$$

3.41

$$f(E) = \left[1 + \exp\left(\frac{E - E_F}{kT}\right)\right]^{-1}$$

so

$$\frac{df(E)}{dE} = (-1)\left[1 + \exp\left(\frac{E - E_F}{kT}\right)\right]^{-2} \times \left(\frac{1}{kT}\right)\exp\left(\frac{E - E_F}{kT}\right)$$

or

$$\frac{df(E)}{dE} = \frac{\frac{-1}{kT}\exp\left(\frac{E - E_F}{kT}\right)}{\left[1 + \exp\left(\frac{E - E_F}{kT}\right)\right]^2}$$

(a) $T = 0$, For

$$E < E_F \Rightarrow \exp(-\infty) = 0 \Rightarrow \frac{df}{dE} = 0$$

$$E > E_F \Rightarrow \exp(+\infty) = +\infty \Rightarrow \frac{df}{dE} = 0$$

$$\text{At } E = E_F \Rightarrow \frac{df}{dE} \to -\infty$$

3.42
(a) At $E = E_{midgap}$,

$$f(E) = \frac{1}{1 + \exp\left(\frac{E - E_F}{kT}\right)} = \frac{1}{1 + \exp\left(\frac{E_g}{2kT}\right)}$$

Si: $E_g = 1.12\ eV$,

$$f(E) = \frac{1}{1 + \exp\left[\frac{1.12}{2(0.0259)}\right]}$$

or

$$\underline{f(E) = 4.07x10^{-10}}$$

Ge: $E_g = 0.66\ eV$,

$$f(E) = \frac{1}{1 + \exp\left[\frac{0.66}{2(0.0259)}\right]}$$

or

$$\underline{f(E) = 2.93x10^{-6}}$$

GaAs: $E_g = 1.42\ eV$,

$$f(E) = \frac{1}{1 + \exp\left[\frac{1.42}{2(0.0259)}\right]}$$

or

$$\underline{f(E) = 1.24x10^{-12}}$$

(b)
Using results of Problem 3.35, the answers to part (b) are exactly the same as those given in part (a).

3.43

$$f(E) = 10^{-6} = \frac{1}{1 + \exp\left(\frac{0.55}{kT}\right)}$$

Then

$$1 + \exp\left(\frac{0.55}{kT}\right) = \frac{1}{10^{-6}} = 10^{+6} \Rightarrow$$

$$\exp\left(\frac{0.55}{kT}\right) \approx 10^{+6} \Rightarrow \left(\frac{0.55}{kT}\right) = \ln\left(10^{6}\right)$$

or

$$kT = \frac{0.55}{\ln\left(10^{6}\right)} \Rightarrow \underline{T = 461K}$$

3.44

At $E = E_2$, $f(E_2) = 0.05$
So

$$0.05 = \frac{1}{1 + \exp\left(\frac{E_2 - E_F}{kT}\right)}$$

Then

$$\frac{E_2 - E_F}{kT} = \ln(19)$$

By symmetry, at $E = E_1$, $1 - f(E_1) = 0.05$,
So

$$\frac{E_F - E_1}{kT} = \ln(19)$$

Then

$$\frac{E_2 - E_1}{kT} = 2\ln(19)$$

(a)
At $T = 300K$, $kT = 0.0259\ eV$

$$E_2 - E_1 = \Delta E = (0.0259)(2)\ln(19) \Rightarrow$$
$$\underline{\Delta E = 0.1525\ eV}$$

(b)
At $T = 500K$, $kT = 0.04317\ eV$

$$E_2 - E_1 = \Delta E = (0.04317)(2)\ln(19) \Rightarrow$$
$$\underline{\Delta E = 0.254\ eV}$$